Maths: One Year Course

Dr. Brendan Williamson

PUBLISHED BY INSTITUTE OF EDUCATION

BRENDANWILLIAMSON.IE

First printing, August 2024

Contents

Introduction . . . xiii

I Numbers

1 Numbers . . . 3

1.1 Different Types of Numbers . . . 3
1.2 Factors, Multiples & Primes . . . 4
1.3 The LCM and HCF . . . 7
1.4 Fractions and Decimals . . . 14
1.5 Rounding and Significant Figures . . . 15
1.6 Inequalities . . . 18
1.7 Interval Notation . . . 27
1.8 Scientific Notation . . . 28
1.9 Manipulating Surds . . . 35
1.10 A Note on Proofs . . . 38
1.11 Proving $\sqrt{2}$ is Irrational . . . 38
1.12 Summary . . . 40
1.13 Homework . . . 41
1.14 Homework Solutions . . . 46
1.15 Harder Problems . . . 50
1.16 Harder Problems Solutions . . . 51
1.17 Revision . . . 54
1.18 Revision Solutions . . . 59

II **Algebra I**

2 Brackets & Factorising ... 65
2.1 Some Terms ... 65
2.2 Adding and Multiplying Terms ... 66
2.3 Expanding Brackets ... 67
2.4 Expanding Multiple Brackets ... 68
2.5 Binomial Expansions ... 72
2.6 Factors of a Term ... 76
2.7 Highest Common Factors of Terms ... 78
2.8 Factorising Using the HCF ... 80
2.9 Factorising Four Terms By Grouping ... 82
2.9.1 $++++$ & $+-+-$ Expressions ... 82
2.9.2 $++--$ & $+--+$ Expressions ... 84
2.9.3 1 and -1 as a HCF ... 85
2.9.4 Rearranging Terms ... 86
2.9.5 Deciding Which Approach to Take ... 86
2.10 Factorising Quadratics ... 87
2.10.1 Of the Form x^2+bx+c ... 87
2.10.2 Of the Form ax^2+bx+c, $a\neq 1$ ... 91
2.10.3 Of the form ax^2+bx+c, when $b=0$ or $c=0$ ... 93
2.11 Difference of Two Squares ... 93
2.12 Sum and Difference of Two Cubes ... 94
2.13 Choosing Between Factorising Methods ... 96
2.14 Summary ... 97
2.15 Homework ... 98
2.16 Homework Solutions ... 100
2.17 Harder Problems ... 102
2.18 Harder Problems Solutions ... 103
2.19 Revision ... 105
2.20 Revision Solutions ... 107

3 Fractions ... 109
3.1 Some Terms and Prerequesites ... 109
3.2 Adding/Subtracting Fractions With Constant Denominator ... 111
3.3 Adding/Subtracting Fractions With Algebraic Denominator ... 114
3.4 More Advanced Addition/Subtraction of Fractions ... 119
3.5 Simplifying Fractions, and Polynomial Long Division ... 122
3.6 Polynomial Long Division with Non-Zero Remainders ... 127
3.7 Complex Fractions ... 129
3.8 Rationalising the Denominator ... 131
3.9 Summary ... 135
3.10 Homework ... 136

3.11 Homework Solutions . . . 138
3.12 Harder Problems . . . 140
3.13 Harder Problems Solutions . . . 141
3.14 Revision . . . 144
3.15 Revision Solutions . . . 146

III Functions

4 Functions . . . 149
4.1 A Visual Representation of Expressions . . . 149
4.2 Some Notation and Definitions . . . 149
4.3 Graphing General Functions . . . 152
4.4 Manipulating Graphs . . . 155
4.5 Equations and Inequalities Involving Functions . . . 168
4.6 Roots and y-intercepts . . . 178
4.7 Injective, Surjective and Bijective Functions . . . 179
4.8 Composition of Functions . . . 181
4.9 Invertible Functions . . . 184
4.10 Summary . . . 190
4.11 Homework . . . 191
4.12 Homework Solutions . . . 200
4.13 Harder Problems . . . 207
4.14 Harder Problems Solutions . . . 208
4.15 Revision . . . 210
4.16 Revision Solutions . . . 221

IV Algebra II

5 Equations . . . 231
5.1 Solutions to Equations, and Manipulating Equations . . . 231
5.2 Linear Equations . . . 233
5.3 Manipulation of Formulae . . . 235
5.3.1 Unpacking to Solve for x . . . 235
5.3.2 Solving for x in general . . . 237
5.3.3 Writing x in Terms of Other Variables . . . 242
5.4 Quadratic Equations . . . 244
5.4.1 Solving by Factorising . . . 244
5.4.2 Solving Using the Quadratic Formula . . . 248
5.4.3 The Number of Solutions . . . 250
5.5 Absolute Value Equations . . . 252
5.6 Surd Equations . . . 254
5.7 Equations With Fractions . . . 260

5.8 Simultaneous Linear Equations in Two Variables 265
5.9 Simultaneous Non-Linear Equations 270
5.10 Simultaneous Linear Equations in Three Variables 274
5.11 Relation to Functions 280
5.12 Summary 280
5.13 Homework 281
5.14 Homework Solutions 284
5.15 Harder Problems 286
5.16 Harder Problems Solutions 287
5.17 Revision 289
5.18 Revision Solutions 292

6 Further Polynomials 295
6.1 Quadratics and Beyond 295
6.2 The Factor Theorem 295
6.3 Undetermined Coefficients 296
6.4 Completed Square Form and Graphing Quadratic Functions 303
6.5 Undetermined Coefficients: An Alternative to Long Division 313
6.6 Cubic Equations 317
6.7 Graphing Cubic Functions 323
6.8 Summary 328
6.9 Homework 329
6.10 Homework Solutions 331
6.11 Harder Problems 333
6.12 Harder Problems Solutions 334
6.13 Revision 336
6.14 Revision Solutions 337

7 Inequalities 339
7.1 Solution Sets and a General Approach 339
7.2 Linear Inequalities 340
7.3 Specifying Number Types, and Writing Solutions on Number Lines 343
7.4 Absolute Value Inequalities 344
7.5 Quadratic Inequalities 346
7.6 Rational Inequalities 349
7.7 Abstract Inequalities 352
7.8 Relation to Functions 354
7.9 Summary 354
7.10 Homework 355
7.11 Homework Solutions 356
7.12 Harder Problems 358
7.13 Harder Problems Solutions 359

7.14 Revision . . . 361
7.15 Revision Solutions . . . 362

V Indices and Logarithms

8 Indices and Logarithms . . . 367
8.1 Powers of x, and x as a Power . . . 367
8.2 Integer Rules of Indices . . . 367
8.3 Fractional Rules of Indices . . . 371
8.4 Combining Indices with Multiplication/Division . . . 373
8.5 Simplifying Expressions Involving Indices . . . 377
8.6 Solving Equations Involving Indices . . . 383
8.7 Logarithms . . . 386
8.8 Logarithm Rules . . . 388
8.9 The Exponential Function and Natural Logarithm . . . 393
8.10 Graphing Indices and Logarithm Functions . . . 396
8.11 Exponentials and Logarithms in Inequalities . . . 401
8.12 Hidden Quadratics . . . 403
8.13 A Return to the Binomial Theorem . . . 408
8.14 Summary . . . 412
8.15 Homework . . . 413
8.16 Homework Solutions . . . 417
8.17 Harder Problems . . . 420
8.18 Harder Problems Solutions . . . 421
8.19 Revision . . . 425
8.20 Revision Solutions . . . 427

VI Shape & Measure

9 Geometry . . . 431
9.1 Angles & Parallel Lines . . . 431
9.2 Triangles . . . 434
9.3 Quadrilaterals & Parallelograms . . . 441
9.4 Circles . . . 444
9.5 Proofs . . . 449
9.6 General Geometry Problems . . . 456
9.7 Summary . . . 463
9.8 Homework . . . 464
9.9 Homework Solutions . . . 469
9.10 Harder Problems . . . 470
9.11 Harder Problems Solutions . . . 471

9.12 **Revision** 472
9.13 **Revision Solutions** 478

10 Constructions 479
10.1 **Notes on the Syllabus** 479
10.2 **Junior Cert Constructions** 480
10.3 **Leaving Cert Constructions** 494
10.4 **Summary** 507

11 Trigonometry 509
11.1 **Right Angled Triangles** 509
11.2 **Special Angles and the Range of the Trigonometric Functions** 516
11.3 **The Unit Circle** 520
11.4 **Trigonometric Equations 1** 525
11.5 **The Sine and Cosine Rule** 527
11.6 **Choosing Between the Sine and Cosine Rule** 540
11.7 **Problems Involving Isosceles Triangles** 541
11.8 **The Area of a Triangle** 542
11.9 **Problems With Multiple Triangles** 546
11.10 **Three Dimensional Problems** 551
11.11 **Angles outside of $0°$ to $360°$** 559
11.12 **Trigonometric Equations 2** 560
11.13 **Radians** 563
11.14 **The Graphs of the Trigonometric Functions** 570
11.15 **Modelling with Trigonometric Functions** 575
11.16 **Formulae 1-24, and $\sec$, $\csc$ and $\cot$** 586
11.17 **Application of Trigonometric Formulae** 589
11.18 **Required Proofs** 596
11.19 **Summary** 602
11.20 **Homework** 603
11.21 **Homework Solutions** 618
11.22 **Harder Problems** 622
11.23 **Harder Problems Solutions** 623
11.24 **Revision** 625
11.25 **Revision Solutions** 638

12 Area, Volume & Surface Area 639
12.1 **Formulae** 639
12.2 **Applying the Formulae** 642
12.2.1 Parallelograms & Trapezoids 642
12.2.2 Arcs, Sectors and Segments 644
12.2.3 Cylinders and Other Prisms 646
12.2.4 Spheres 647

12.2.5 Cones 647
12.2.6 Frusta 651
12.2.7 Pyramids 654
12.2.8 The Trapezoid Rule 655
12.3 Combining Shapes, and Harder Problems 656
12.4 Enlargements 658
12.5 Summary 660
12.6 Homework 661
12.7 Homework Solutions 664
12.8 Harder Problems 665
12.9 Revision 666

VII Coordinate Geometry

13 Coordinate Geometry of the Line 669
13.1 Revision of the Coordinate Plane 669
13.2 Revision of Coordinate Geometry Formulae 671
13.3 More on the Slope 678
13.4 More on the Equation of the Line 683
13.4.1 Checking if a Point is on a Line 683
13.4.2 The x and y Intercepts 684
13.4.3 Horizontal and Vertical Lines 686
13.4.4 Different Forms of the Equation of the Line, and Finding the Slope 687
13.5 Intersection of Two Lines 689
13.6 Parallel and Perpendicular Lines 691
13.7 Finding the Equation of the Line with Different Information 694
13.8 Dividing a Line in a Given Ratio 696
13.9 Angle of Inclination of a Line 699
13.10 Angle Between Two Lines 702
13.11 Perpendicular Distance From a Point to a Line 706
13.12 Area of a Triangle 709
13.13 The Centroid of a Triangle 712
13.14 The Orthocentre of a Triangle 714
13.15 Summary 717
13.16 Homework 718
13.17 Homework Solutions 724
13.18 Harder Problems 728
13.19 Harder Problems Solutions 729
13.20 Revision 734
13.21 Revision Solutions 740

14 Coordinate Geometry of the Circle . . . 745
14.1 The Equation of the Circle . . . 745
14.2 Intersection of Lines and Circles . . . 749
14.3 Tangents to a Circle . . . 750
14.4 Touching Circles . . . 756
14.5 Finding the Equation of a Circle From Information . . . 757
14.6 Summary . . . 763
14.7 Homework . . . 764
14.8 Homework Solutions . . . 766
14.9 Harder Problems . . . 772
14.10 Harder Problems Solutions . . . 774
14.11 Revision . . . 776
14.12 Revision Solutions . . . 778

VIII Calculus

15 Differentiation . . . 785
15.1 A Reminder about Slopes . . . 785
15.2 The Rate of Change of a Function . . . 786
15.3 Limits . . . 789
15.4 Derivatives of a Function: First Principles . . . 794
15.5 Differentiation by Rule . . . 796
15.6 Slope and Equation of the Tangent Line . . . 798
15.7 The Product, Quotient and Chain Rule . . . 800
15.8 Second Derivatives . . . 803
15.9 Implicit Differentiation . . . 804
15.10 Turning Points of Curves, & Points of Inflection . . . 806
15.11 Sketching Graphs . . . 812
15.12 Undetermined Coefficients Revisited . . . 815
15.13 Max/Min Problems . . . 816
15.14 Rates of Change . . . 818
15.15 Acceleration, Velocity and Displacement . . . 819
15.16 Related Rates . . . 822
15.17 Summary . . . 825
15.18 Homework . . . 826
15.19 Homework Solutions . . . 832
15.20 Harder Problems . . . 836
15.21 Harder Problems Solutions . . . 837
15.22 Revision . . . 844
15.23 Revision Solutions . . . 849

16 Integration . . . 855
16.1 Antidifferentiating: The Indefinite Integral . . . 855
16.2 Integration of Elementary Functions . . . 856
16.3 Returning to Displacement, Velocity and Acceleration . . . 861
16.4 Definite Integrals . . . 862
16.5 Area Under A Curve . . . 864
16.6 Area Between Curves . . . 870
16.7 The Trapezoid Rule and Graphs . . . 878
16.8 Average Value of a Function . . . 880
16.9 Advanced Trigonometric Integrals . . . 883
16.10 Definite vs Indefinite Integrals . . . 884
16.11 Summary . . . 884
16.12 Homework . . . 885
16.13 Homework Solutions . . . 890
16.14 Harder Problems . . . 892
16.15 Harder Problems Solutions . . . 895
16.16 Revision . . . 899
16.17 Revision Solutions . . . 903

IX Discrete & Practical Maths

17 Sequences & Series . . . 907
17.1 Notation and Definitions . . . 907
17.2 Recurrence Relations . . . 908
17.3 Arithmetic Sequences . . . 910
17.4 Arithmetic Series . . . 916
17.5 Abstract Arithmetic Sequence Problems . . . 921
17.6 Geometric Sequences . . . 922
17.7 Geometric Series . . . 929
17.8 Infinite Geometric Sequences . . . 932
17.9 Abstract Geometric Sequence Problems . . . 936
17.10 Recurring Decimals . . . 937
17.11 Quadratic Sequences . . . 939
17.12 Cubic Sequences . . . 943
17.13 Harder Problems . . . 952
17.14 Harder Problems Solutions . . . 954
17.15 Revision . . . 957
17.16 Revision Solutions . . . 958

18 Applied Arithmetic . . . 959
18.1 Mathematical Miscellanea . . . 959

18.2 Ratios . . . 959
18.3 Percentages . . . 961
18.4 Error, Tolerance and Rounding . . . 962
18.5 Units, Conversion and Per Unit Cost . . . 965
18.6 General Compound Units . . . 967
18.7 Distance, Speed and Time . . . 968
18.8 Mark Up & Margin . . . 970
18.9 VAT . . . 972
18.10 Income Deductions . . . 973
18.11 Summary . . . 975
18.12 Homework . . . 976
18.13 Homework Solutions . . . 979
18.14 Harder Problems . . . 980
18.15 Revision . . . 981
18.16 Revision Solutions . . . 982

19 Financial Mathematics . . . 983
19.1 Interest Rates . . . 983
19.2 Depreciation . . . 987
19.3 Cash Flows . . . 988
19.4 Non-annual Interest Rates . . . 989
19.5 Annuities . . . 991
19.6 Perpetuities . . . 995
19.7 Summary . . . 996
19.8 Homework . . . 997
19.9 Homework Solutions . . . 1000
19.10 Harder Problems . . . 1002
19.11 Harder Problems Solutions . . . 1003
19.12 Revision . . . 1005
19.13 Revision Solutions . . . 1007

20 Proof by Induction . . . 1009
20.1 Under Reconstruction . . . 1009

X Complex Numbers

21 Complex Numbers . . . 1013
21.1 Under Reconstruction . . . 1013

XI Probability & Statistics

22 Counting 1017

22.1 Under Reconstruction 1017

23 Probability 1019

23.1 Under Reconstruction 1019

24 Statistics 1021

24.1 Under Reconstruction 1021

Introduction

This book is designed to be read as a single document, in the order in which the chapters appear. In each chapter questions, examples, definitions etc. are on a common numbering system, meaning that Definition 4.7 is the 7th text box in Chapter 4. Similarly figures are on a separate common numbering system, so that Figure 7.12 is the 12th picture in Chapter 7. This book also appears in pdf format on Moodle, in which all references are hyperlinked for easier navigation.

At the end of each chapter is a summary, which explains how the content of the chapter fits into the Leaving Cert syllabus and maps onto past (and possibly future) exam questions. There is also a set of homework problems (separated by section) with solutions provided at the end of the chapter. This is to be completed as we work through the chapter. There is also a revision section at the end of each chapter for students re-reading the chapter or preparing for a test, as well as a section with harder problems for the more ambitious student.

This book is comprehensive, assuming no knowledge of Maths beyond the basics of Junior Cycle and so you don't need any other book or notes to study for the course. It is also designed so that it can be read as revision weeks or months after we first cover them. As such you shouldn't need to spend much time taking notes and can instead concentrate on the class. However you may want to take notes occasionally if there is something mentioned in class that isn't covered clearly in the book.

If you have any questions you can reach me at bwilliamson@instituteofeducation.ie.

V Indices and Logarithms

8 Indices and Logarithms 367

8.1 Powers of x, and x as a Power 367
8.2 Integer Rules of Indices 367
8.3 Fractional Rules of Indices 371
8.4 Combining Indices with Multiplication/Division 373
8.5 Simplifying Expressions Involving Indices 377
8.6 Solving Equations Involving Indices 383
8.7 Logarithms 386
8.8 Logarithm Rules 388
8.9 The Exponential Function and Natural Logarithm 393
8.10 Graphing Indices and Logarithm Functions 396
8.11 Exponentials and Logarithms in Inequalities 401
8.12 Hidden Quadratics 403
8.13 A Return to the Binomial Theorem 408
8.14 Summary 412
8.15 Homework 413
8.16 Homework Solutions 417
8.17 Harder Problems 420
8.18 Harder Problems Solutions 421
8.19 Revision 425
8.20 Revision Solutions 427

8. Indices and Logarithms

8.1 Powers of x, and x as a Power

So far this year we have avoided an indepth study of powers, or indices. Here will study how powers behave, for example if we want to simplify x^3x^4, as well as study how expressions such as 2^x or 3^x behave as x varies. Throughout this chapter we will use the following terms.

Definition 8.1 For the term x^n, we say that x is the **base** and n is the **index** (plural indices) or **power**.

8.2 Integer Rules of Indices

When multiplying powers of x there are certain rules we can use so that we don't need ro rely on first principles. For example,

$$\begin{aligned} x^3x^4 &= \underbrace{x\times x\times x}_{3}\times\underbrace{x\times x\times x\times x}_{4} \\ &= x^7. \end{aligned}$$

Similarly,

$$\begin{aligned} x^4x^6 &= \underbrace{x\times x\times x\times x}_{4}\times\underbrace{x\times x\times x\times x\times x\times x}_{6} \\ &= x^{10}. \end{aligned}$$

Noticing the pattern here, we have our first rule.

Rule 8.2 For any $m,n\in\mathbb{N}$,

$$x^mx^n = x^{m+n}.$$

Question 8.3 Write the following expressions as x^n for some $n \in \mathbb{N}$.

$$x^5x^4$$
$$x^1x^7$$
$$x^2x^9$$
$$x^3x^2x^4$$

As powers are just repeated multiplication, see that

$$\begin{aligned}\left(x^4\right)^3 &= \underbrace{x^4x^4x^4}_{3} \\ &= x^{12}.\end{aligned}$$

Similarly

$$\begin{aligned}\left(x^5\right)^6 &= \underbrace{x^5x^5x^5x^5x^5x^5}_{6} \\ &= x^{30}.\end{aligned}$$

Noticing the pattern, the powers get multiplied to give the new power of x. This gives us our second rule which we add to the first.

Rule 8.4 For any $m,n \in \mathbb{N}$,

$$x^mx^n = x^{m+n},$$
$$(*) \quad (x^m)^n = x^{mn}.$$

Question 8.5 Write the following expressions as x^n for some $n \in \mathbb{N}$.

$$\left(x^3\right)^2$$
$$\left(x^3\right)^6$$
$$\left(x^2\right)^5$$
$$x^7\left(x^3\right)^2$$
$$\left(x^3x^2\right)^4$$

We can apply a similar logic to fractions as we do to products. See that

$$\begin{aligned}\frac{x^5}{x^2} &= \frac{\cancel{x \times x} \times x \times x \times x}{\cancel{x \times x}} \\ &= x \times x \times x \\ &= x^3.\end{aligned}$$

Similarly,

$$\begin{aligned}\frac{x^8}{x^3} &= \frac{\cancel{x \times x \times x} \times x \times x \times x \times x \times x}{\cancel{x \times x \times x}} \\ &= x \times x \times x \times x \times x \\ &= x^5.\end{aligned}$$

Noticing the pattern, we subtract the power in the denominator from the power in the numerator to give us the new power of x. This gives our third rule to add to our collection.

Rule 8.6 For any $m, n \in \mathbb{N}$,

$$x^m x^n = x^{m+n},$$

$$(*) \quad \frac{x^m}{x^n} = x^{m-n} \quad \text{if } m > n,$$

$$(x^m)^n = x^{mn}.$$

Question 8.7 Write the following expressions as x^n for some $n \in \mathbb{N}$.

$$\frac{x^6}{x^3}$$

$$\frac{x^7}{x^4}$$

$$\frac{x^9}{x^2}$$

$$\frac{x^5}{x}$$

$$\frac{x^8}{x^3x^2}$$

$$\frac{x^2x^5}{x^3}$$

What if we apply the last rule when we don't have $m > n$? If we did, we would get weird answers. See that using the last rule gives us

$$\frac{x^4}{x^4} = x^0,$$

$$\frac{x^4}{x^5} = x^{-1},$$

$$\frac{x^4}{x^7} = x^{-3}.$$

So what are x^0, x^{-1}, x^{-3}? To understand non-positive powers of x consider the following progression.

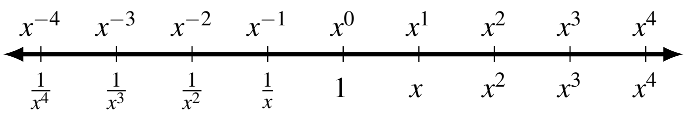


Figure 8.1

On the top row, when we move to the right we increase the power by 1. On the bottom row when we move to the right we multiply by x. In the case of positive powers at least, we should understand that "increasing the power by one" and "multiplying by x" are the same. Similarly on the top row, when we move to the left we decrease the power by 1. On the bottom row when we move to the left we divide by x. Again, in the case of positive powers at least, we should understand that "decreasing the power by one" and "dividing by x" are the same. What we are saying here is that we can continue this pattern to the left in Figure 8.1 to understand negative powers of x.

Another way of considering negative powers of x is as follows. Rather than thinking of x^3 as "three x's multiplied together", think of it as being equal to what you get when you "start with 1 and multiply by x, 3 times". Therefore x^{-3} is what you get when you "start with 1 and multiply by x, -3 times". While this sounds odd, in maths doing something a negative number of times is like doing its opposite that positive number of times. Therefore "multiply by x, -3 times" is the same as "divide by x, 3 times". Therefore $x^{-3} = \frac{1}{x^3}$.

This definition also agrees with our rule $\frac{x^m}{x^n} = x^{m-n}$ for $m \leq n$. Calculating the fractions below using the rule (on the right) and more traditional algebra (on the left) we see that both answers line up appear together on the number line in all cases.

$$1 = \frac{x^4}{x^4} = x^0,$$

$$\frac{1}{x} = \frac{x^4}{x^5} = x^{-1},$$

$$\frac{1}{x^3} = \frac{x^4}{x^7} = x^{-3}.$$

This gives us the following update to our list of rules.

Rule 8.8 For any $m, n \in \mathbb{N}$,

$$x^m x^n = x^{m+n},$$

$$(*) \quad \frac{x^m}{x^n} = x^{m-n},$$

$$(x^m)^n = x^{mn},$$

$$(*) \quad x^0 = 1,$$

$$(*) \quad x^{-n} = \frac{1}{x^n}.$$

Question 8.9 Write the following expressions as x^n for some $n \in \mathbb{Z}$

$$\frac{x^2}{x^5}$$

$$\frac{x^2}{x^3}$$

$$\frac{x^3}{x^5}$$

$$\frac{x^5}{x^5}$$

Question 8.10 Write the following as $\frac{1}{x^n}$ for some $n \in \mathbb{N}$.

$$x^{-1}$$
$$x^{-3}$$
$$x^{-7}$$

Question 8.11 Write the following without indices.

$$2^{-2}$$
$$4^{-3}$$
$$5^{-1}$$
$$2^{-4}$$
$$8^{0}$$

8.3 Fractional Rules of Indices

When studying the difference of two squares and the difference/sum of two cubes we glanced at the idea of taking the square or cube root of powers of x. For example, when factorising $x^4 - y^6$ we had

$$\begin{aligned} x^4 - y^6 &= \left(x^2\right)^2 - (y^3)^2 \\ &= (x^2 - y^3)(x^2 + y^3). \end{aligned}$$

The reason x^2 went inside the bracket on the first line was because it was the square root of x^4, i.e.

$$\sqrt{x^4} = x^2.$$

Similarly

$$\sqrt{y^6} = y^3.$$

In general, it would seem that

$$\sqrt{x^m} = x^{\frac{m}{2}}.$$

Similarly when factorising $x^6 - y^9$ using the difference of two cubes we had

$$\begin{aligned} x^6 - y^9 &= (x^2)^3 - (y^3)^3 \\ &= (x^2 - y^3)((x^2)^2 + x^2y^3 + (y^3)^2) \end{aligned}$$

precisely because

$$\begin{aligned} \sqrt[3]{x^6} &= x^2, \\ \sqrt[3]{y^9} &= y^3. \end{aligned}$$

In general,

$$\sqrt[3]{x^m} = x^{\frac{m}{3}}.$$

Following this pattern, we could say that

$$\sqrt[n]{x^m} = x^{\frac{m}{n}},$$

at least when $\frac{m}{n}$ is an integer. But much like the rule

$$\frac{x^m}{x^n} = x^{m-n}$$

leading us to define negative powers of x, we will also consider fractional powers of x that arise from this formula. For example,

$$\sqrt{x} = x^{\frac{1}{2}},$$
$$\sqrt[3]{x^7} = x^{\frac{7}{3}}.$$

This gives us another rule to add to our list.

Rule 8.12 For any $m, n \in \mathbb{N}$,

$$x^m x^n = x^{m+n},$$
$$\frac{x^m}{x^n} = x^{m-n},$$
$$(x^m)^n = x^{mn},$$
$$x^0 = 1,$$
$$x^{-n} = \frac{1}{x^n},$$
$$(*) \qquad \sqrt[n]{x^m} = x^{\frac{m}{n}}.$$

We will quite often use this rule in reverse. For example, if asked to calculate $8^{\frac{2}{3}}$ we understand that

$$\begin{aligned} 8^{\frac{2}{3}} &= \sqrt[3]{8^2} \\ &= \sqrt[3]{64} \\ &= 4. \end{aligned}$$

It also holds that we can take the root first and then apply the power;

$$\begin{aligned} 8^{\frac{2}{3}} &= \sqrt[3]{8}^2 \\ &= 2^2 \\ &= 4, \end{aligned}$$

giving us yet another update to our list of rules.

Rule 8.13 For any $m,n \in \mathbb{N}$,

$$x^m x^n = x^{m+n},$$
$$\frac{x^m}{x^n} = x^{m-n},$$
$$(x^m)^n = x^{mn},$$
$$x^0 = 1,$$
$$x^{-n} = \frac{1}{x^n},$$
$$(*) \quad \sqrt[n]{x^m} = \sqrt[n]{x}^m = x^{\frac{m}{n}}.$$

Question 8.14 Write the following in the form $\sqrt[n]{x^m}$ or $\sqrt[n]{x}^m$ for some $m,n \in \mathbb{N}$.

$$x^{\frac{7}{3}}$$
$$x^{\frac{2}{5}}$$
$$x^{\frac{1}{4}}$$

Question 8.15 Calculate the following without a calculator.

$$9^{\frac{1}{2}}$$
$$8^{\frac{1}{3}}$$
$$27^{\frac{2}{3}}$$
$$4^{\frac{5}{2}}$$

8.4 Combining Indices with Multiplication/Division

This year we have already used the fact that, for example,

$$(3x)^2 = 9x^2,$$

i.e. in squaring the $3x$ we square the 3 and square the x. This is true in general for any factors and for any power; for any a, b and any power n we

$$(ab)^n = a^n b^n.$$

Example 8.16 Get rid of the brackets in the following expression.

$$(2y)^3$$

$$(2y)^3 = 8y^3.$$

Example 8.17 Get rid of the brackets in the following expression.

$$\left(4x^3\right)^2$$

$$\begin{aligned}\left(4x^3\right)^2 &= 4^2\left(x^3\right)^2\\ &= 16x^6.\end{aligned}$$

There is a similar rule for fractions;

$$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}.$$

Example 8.18 Get rid of the brackets in the following expression.

$$\left(\frac{3}{4}\right)^2$$

$$\left(\frac{3}{4}\right)^2 = \frac{9}{16}.$$

Example 8.19 Get rid of the brackets in the following expression.

$$\left(\frac{x}{2}\right)^3$$

$$\left(\frac{x}{2}\right)^3 = \frac{x^3}{8}.$$

These two final rules are added to our list of rules below.

Rule 8.20 — Rules of Indices (pg. 21 of *Formulae & Tables*). For any $m, n \in \mathbb{N}$,

$$\begin{aligned}x^m x^n &= x^{m+n},\\ \frac{x^m}{x^n} &= x^{m-n},\\ (x^m)^n &= x^{mn},\\ x^0 &= 1,\\ x^{-n} &= \frac{1}{x^n},\\ \sqrt[n]{x^m} &= \sqrt[n]{x}^m = x^{\frac{m}{n}},\\ (*) \quad (xy)^n &= x^n y^n,\\ (*) \quad \left(\frac{x}{y}\right)^n &= \frac{x^n}{y^n}.\end{aligned}$$

These rules appear in the following form on page 21 of *Formulae and Tables*.

Séana agus logartaim		Indices and logarithms
$a^p a^q = a^{p+q}$	$\log_a(xy) = \log_a x + \log_a y$	$a^x = y \quad \Leftrightarrow \quad \log_a y = x$
$\dfrac{a^p}{a^q} = a^{p-q}$	$\log_a\left(\dfrac{x}{y}\right) = \log_a x - \log_a y$	$\log_a\left(a^x\right) = x$
$\left(a^p\right)^q = a^{pq}$	$\log_a\left(x^q\right) = q\log_a x$	$a^{\log_a x} = x$
$a^0 = 1$	$\log_a 1 = 0$	
$a^{-p} = \dfrac{1}{a^p}$	$\log_a\left(\dfrac{1}{x}\right) = -\log_a x$	$\log_b x = \dfrac{\log_a x}{\log_a b}$
$a^{\frac{1}{q}} = \sqrt[q]{a}$		
$a^{\frac{p}{q}} = \sqrt[q]{a^p} = \left(\sqrt[q]{a}\right)^p$		
$(ab)^p = a^p b^p$		
$\left(\dfrac{a}{b}\right)^p = \dfrac{a^p}{b^p}$		

- 21 -

Figure 8.2

Question 8.21 Get rid of the brackets in the following expressions.

$$(5x)^2$$
$$(3y)^3$$
$$(4z^2)^2$$
$$\left(\frac{x}{3}\right)^2$$
$$\left(\frac{2}{y}\right)^3$$
$$\left(\frac{x}{y^2}\right)^4$$
$$\left(\frac{x}{3y}\right)^2$$

These last two rules also apply to non-integer powers, which also means they apply to roots.

Example 8.22 Get rid of the brackets in the following expression.

$$\left(\frac{9}{4}\right)^{\frac{3}{2}}$$

$$\begin{aligned}\left(\frac{9}{4}\right)^{\frac{3}{2}} &= \frac{9^{\frac{3}{2}}}{4^{\frac{3}{2}}}\\ &= \frac{\sqrt{9}^3}{\sqrt{4}^3}\\ &= \frac{3^3}{2^3}\\ &= \frac{27}{8}.\end{aligned}$$

Example 8.23 Write the following expression without a root.

$$\sqrt{\frac{64}{49}}.$$

We could say that

$$\begin{aligned}\sqrt{\frac{64}{49}} &= \left(\frac{64}{49}\right)^{\frac{1}{2}}\\ &= \frac{64^{\frac{1}{2}}}{49^{\frac{1}{2}}}\\ &= \frac{\sqrt{64}}{\sqrt{49}}\\ &= \frac{8}{7}.\end{aligned}$$

However this is unnecessarily long winded. We can simply understand that roots act like powers and say

$$\begin{aligned}\sqrt{\frac{64}{49}} &= \frac{\sqrt{64}}{\sqrt{49}}\\ &= \frac{8}{7}.\end{aligned}$$

Example 8.24 Write the following expression without a root.

$$\sqrt[3]{8x^6}$$

$$\sqrt[3]{8x^6} = \sqrt[3]{8}\sqrt[3]{x^6}$$
$$= 2x^2.$$

Question 8.25 Write the following expressions without a root.

$$\sqrt{4x^2}$$
$$\sqrt[3]{27y^6}$$
$$\sqrt{\frac{x^2}{y^4}}$$
$$\sqrt{\frac{16}{9}}$$
$$\sqrt[3]{\frac{27}{8}}$$
$$\sqrt[3]{\frac{x^6}{8}}.$$

8.5 Simplifying Expressions Involving Indices

Now that we have all of our rules of indices we can combine them to simplify expressions involving powers.

Example 8.26 Write the expression

$$\frac{(x^2x^3)^5}{x^9}$$

as x^n for some $n \in \mathbb{Q}$.

$$\frac{(x^2x^3)^5}{x^9} = \frac{(x^5)^5}{x^9}$$
$$= \frac{x^{25}}{x^9}$$
$$= x^{16}.$$

Sometimes these expressions will be include roots that are not written as powers. In that case we need to write them as powers ourselves.

Example 8.27 Write the expression

$$\left(x^2\sqrt{x}\right)^9$$

as x^n for some $n \in \mathbb{Q}$.

$$\begin{aligned}\left(x^2\sqrt{x}\right)^9 &= \left(x^2x^{\frac{1}{2}}\right)^9\\ &= \left(x^{\frac{5}{2}}\right)^9\\ &= x^{\frac{45}{2}}.\end{aligned}$$

Example 8.28 Write the expression

$$\sqrt[3]{\frac{x^5x^4}{x^3}}$$

as x^n for some $n \in \mathbb{Q}$.

$$\begin{aligned}\sqrt[3]{\frac{x^5x^4}{x^3}} &= \left(\frac{x^9}{x^3}\right)^{\frac{1}{3}}\\ &= \left(x^6\right)^{\frac{1}{3}}\\ &= x^2.\end{aligned}$$

Question 8.29 Write the expression

$$\frac{x^2x^3}{x^9}$$

as x^n for some $n \in \mathbb{Q}$.

Question 8.30 Write the expression

$$\frac{x\sqrt[3]{x^5}}{x^2}$$

as x^n for some $n \in \mathbb{Q}$.

Question 8.31 Write the expression

$$\sqrt{(x^2)^3x}$$

as x^n for some $n \in \mathbb{Q}$.

Sometimes in these expressions the base is known but there are variables in some of the powers. In this case it is unwise to calculate the expressions directly; it is better to leave them as powers so that we can simplify the expression.

Example 8.32 Write the expression

$$\frac{2^3 2^x}{2^7}$$

as 2^{ax+b} for some $a, b \in \mathbb{Q}$.

We don't want to replace 2^3 with 8 or 2^7 with 128; we want to keep all terms as a power of 2.

$$\begin{aligned}\frac{2^3 2^x}{2^7} &= \frac{2^{3+x}}{2^7}\\ &= 2^{3+x-7}\\ &= 2^{x-4}.\end{aligned}$$

Example 8.33 Write the expression

$$\left(\frac{\sqrt{3^x}}{3^3}\right)^3$$

as 3^{ax+b} for some $a, b \in \mathbb{Q}$.

$$\begin{aligned}\left(\frac{\sqrt{3^x}}{3^3}\right)^3 &= \left(\frac{(3^x)^{\frac{1}{2}}}{3^3}\right)^3\\ &= \left(\frac{3^{\frac{x}{2}}}{3^3}\right)^3\\ &= \left(3^{\frac{x}{2}-3}\right)^3\\ &= 3^{3\left(\frac{x}{2}-3\right)}\\ &= 3^{\frac{3x}{2}-9}.\end{aligned}$$

Sometimes not all expressions will have the same base to begin with. In this case our first job is to write all expressions as powers of the same base. To help us with this we have the following rule.

Rule 8.34 When simplifying indices expressions involve multiple bases,

1. Write each piece of the expression as a power of the same base. Usually the base will be the smallest of the bases shown.
2. Use the Rules of Indices to write the expression as a single base to a single power.

Example 8.35 Write the expression

$$\frac{2^x 2^7}{4}$$

as 2^{ax+b} for some $a, b \in \mathbb{Q}$.

By Rule 8.34 we should try to write 4 as a power of 2. We can do this; $2^2 = 4$. Then

$$\begin{aligned}\frac{2^x 2^7}{4} &= \frac{2^x 2^7}{2^2} \\ &= \frac{2^{x+7}}{2^2} \\ &= 2^{x+7-2} \\ &= 2^{x+5}.\end{aligned}$$

Example 8.36 Write the expression

$$\frac{\sqrt{3}(3^x)}{27}$$

as 3^{ax+b} for some $a, b \in \mathbb{Q}$.

By Rule 8.34 we should try to write 27 as a power of 3. We can do this; $3^3 = 27$. Then

$$\begin{aligned}\frac{\sqrt{3}(3^x)}{27} &= \frac{3^{\frac{1}{2}}(3^x)}{3^3} \\ &= \frac{3^{\frac{1}{2}+x}}{3^3} \\ &= 3^{\frac{1}{2}+x-3} \\ &= 3^{x-\frac{5}{2}}.\end{aligned}$$

Based on what we have done so far there is no easy way to know that $27 = 3^3$ except by checking (we will solve this problem later). However once we know by Rule 8.34 that we should write every expression as a power of 3, we know that we should write 27 as "3 to the power of something". Then it's just a matter of checking which power of 3 is equal to 27.

Example 8.37 Write the expression

$$\frac{\sqrt{8}(2^x)}{16}$$

as 2^{ax+b} for some $a, b \in \mathbb{Q}$.

$$\begin{aligned}
\frac{\sqrt{8}(2^x)}{16} &= \frac{\sqrt{2^3}(2^x)}{2^4} \\
&= \frac{2^{\frac{3}{2}}2^x}{2^4} \\
&= \frac{2^{\frac{3}{2}+x}}{2^4} \\
&= 2^{\frac{3}{2}+x-4} \\
&= 2^{x-\frac{5}{2}}.
\end{aligned}$$

Question 8.38 Write the expression

$$\frac{2^x}{2^3}$$

as 2^{ax+b} for some $a, b \in \mathbb{Q}$.

Question 8.39 Write the expression

$$\frac{9\sqrt{3^{5x}}}{3^4}$$

as 3^{ax+b} for some $a, b \in \mathbb{Q}$.

Question 8.40 Write the expression

$$\left(\frac{(125^x)\sqrt[4]{5}}{5}\right)^3$$

as 5^{ax+b} for some $a, b \in \mathbb{Q}$.

Question 8.41 Write the expression

$$3^x\sqrt{27}\sqrt[3]{9}$$

as 3^{ax+b} for some $a, b \in \mathbb{Q}$.

More difficult problems of this type will have more than two bases, where it is impossible to easily write the larger as a power of the smaller. In this case we abandon Rule 8.34 and use Rule 8.42.

Rule 8.42 If, when simplifying indices expressions involving multiple bases, it is impossible to write all larger bases as powers of the smaller base, write all bases as powers of an even smaller base not present in the expression.

Example 8.43 Write the expression

$$4^x\sqrt{8}$$

as p^{ax+b} for some $a,b \in \mathbb{Q}$, $p \in \mathbb{N}$.

Here we cannot easily write 8 as a power of 4. However we can write both as a power of 2.

$$\begin{aligned} 4^x\sqrt{8} &= (2^2)^x\sqrt{2^3} \\ &= 2^{2x}2^{\frac{3}{2}} \\ &= 2^{2x+\frac{3}{2}}. \end{aligned}$$

Example 8.44 Write the expression

$$\frac{27^x}{81}$$

as p^{ax+b} for some $a,b \in \mathbb{Q}$, $p \in \mathbb{N}$.

Here both powers can be written as a power of 3.

$$\begin{aligned} \frac{27^x}{81} &= \frac{(3^3)^x}{3^4} \\ &= \frac{3^{3x}}{3^4} \\ &= 3^{3x-4}. \end{aligned}$$

Example 8.45 Write the expression

$$\frac{\sqrt{216}}{36^x}$$

as p^{ax+b} for some $a,b \in \mathbb{Q}$, $p \in \mathbb{N}$.

$$\begin{aligned} \frac{\sqrt{216}}{36^x} &= \frac{\sqrt{6^3}}{(6^2)^x} \\ &= \frac{6^{\frac{3}{2}}}{6^{2x}} \\ &= 6^{\frac{3}{2}-2x}. \end{aligned}$$

Question 8.46 Write the expression

$$\frac{4^x}{\sqrt{8}}$$

as p^{ax+b} for some $a, b \in \mathbb{Q}$, $p \in \mathbb{N}$.

Question 8.47 Write the expression

$$8^x \sqrt[3]{16}$$

as p^{ax+b} for some $a, b \in \mathbb{Q}$, $p \in \mathbb{N}$.

Question 8.48 Write the expression

$$\frac{125}{25^x}$$

as p^{ax+b} for some $a, b \in \mathbb{Q}$, $p \in \mathbb{N}$.

8.6 Solving Equations Involving Indices

There are many equations involving indices, but for now we will focus on equations that contain only one term on each side of the equals sign. Those terms will look like those we simplified in Section 8.5. For example, the equation

$$\frac{2^x}{2^3} = 2^4\sqrt{2}$$

or

$$\frac{9}{3^x} = \frac{27^x}{3}$$

We solve these equations in four steps.

Rule 8.49 To solve equations involving indices,

1. Write every term on both sides of the equals sign as a power of the same base.
2. Simplify both sides of the equals sign so that both sides only have one term which is a base to a certain power.
3. Set the powers on both sides of the equals sign to each other.
4. Solve the resulting equation for x.

Example 8.50 Solve the equation

$$\frac{9}{3^x} = \frac{27^x}{3}.$$

1. Writing every term as a power of 3,

$$\frac{9}{3^x} = \frac{27^x}{3}$$
$$\Rightarrow \frac{3^2}{3^x} = \frac{(3^3)^x}{3^1}.$$

2. Simplifying both sides,

$$\frac{3^2}{3^x} = \frac{(3^3)^x}{3^1}$$
$$\Rightarrow 3^{2-x} = \frac{3^{3x}}{3^1}$$
$$\Rightarrow 3^{2-x} = 3^{3x-1}.$$

3. Setting the two powers equal to each other,

$$2 - x = 3x - 1.$$

4. Solving this equation,

$$2 - x = 3x - 1$$
$$\Rightarrow 3 = 4x$$
$$\Rightarrow \frac{3}{4} = x.$$

Example 8.51 Solve the equation

$$125\sqrt{5} = \sqrt[3]{5^x}.$$

1. Writing every term as a power of 5,

$$125\sqrt{5} = \sqrt[3]{5^x}$$
$$\Rightarrow 5^3 5^{\frac{1}{2}} = 5^{\frac{x}{3}}.$$

2. Simplifying both sides,

$$5^3 5^{\frac{1}{2}} = 5^{\frac{x}{3}}$$
$$\Rightarrow 5^{\frac{7}{2}} = 5^{\frac{x}{3}}.$$

3. Setting the two powers equal to each other,

$$\frac{7}{2} = \frac{x}{3}.$$

4. Solving this equation,

$$\frac{7}{2} = \frac{x}{3}$$
$$\Rightarrow \frac{21}{2} = x.$$

Just like in the previous section, sometimes its not as simple as choosing the smallest base in the equation to be the base for all terms.

Example 8.52 Solve the equation

$$\frac{8}{4^x} = 16.$$

---

1. We can't easily write 8 as a power of 4, so writing every term as a power of 2,

$$\frac{8}{4^x} = 16$$
$$\Rightarrow \frac{2^3}{(2^2)^x} = 2^4.$$

2. Simplifying both sides,

$$\frac{2^3}{(2^2)^x} = 2^4$$
$$\Rightarrow \frac{2^3}{2^{2x}} = 2^4$$
$$\Rightarrow 2^{3-2x} = 2^4.$$

3. Setting the two powers equal to each other,

$$3 - 2x = 4.$$

4. Solving this equation,

$$3 - 2x = 4$$
$$\Rightarrow -1 = 2x$$
$$\Rightarrow -\frac{1}{2} = x.$$

Question 8.53 Solve the following equations.

$$8\sqrt{2} = 4^x$$
$$\frac{6}{\sqrt{6}} = 216^x$$
$$\frac{10^x}{\sqrt{1000}} = 100^{3-x}.$$

8.7 Logarithms

Earlier in this chapter we said "Based on what we have done so far there is no easy way there is no easy way to know that $27 = 3^3$ except by checking". We will now solve that problem.

Definition 8.54 The logarithm function $\log_a x$ (read as "log of x to the base a") finds the solution b to the equation $a^b = x$. In otherwords, $\log_a x$ is the inverse of the a^x function.

This is stated on the top right of page 21 of *Formulae & Tables*.

Séana agus logartaim		Indices and logarithms
$a^p a^q = a^{p+q}$	$\log_a(xy) = \log_a x + \log_a y$	$a^x = y \quad \Leftrightarrow \quad \log_a y = x$
$\frac{a^p}{a^q} = a^{p-q}$	$\log_a\left(\frac{x}{y}\right) = \log_a x - \log_a y$	$\log_a\left(a^x\right) = x$
$\left(a^p\right)^q = a^{pq}$	$\log_a\left(x^q\right) = q\log_a x$	$a^{\log_a x} = x$
$a^0 = 1$	$\log_a 1 = 0$	
$a^{-p} = \frac{1}{a^p}$	$\log_a\left(\frac{1}{x}\right) = -\log_a x$	$\log_b x = \frac{\log_a x}{\log_a b}$
$a^{\frac{1}{q}} = \sqrt[q]{a}$		
$a^{\frac{p}{q}} = \sqrt[q]{a^p} = \left(\sqrt[q]{a}\right)^p$		
$(ab)^p = a^p b^p$		
$\left(\frac{a}{b}\right)^p = \frac{a^p}{b^p}$		

- 21 -

Figure 8.3

The logarithm function can be found on your calculator, usually denoted $\log_{\square}\square$. Both numbers have to be inputted into the calculator in their respective positions.

Example 8.55 Find the solution to $3^x = 243$.

$$\begin{aligned} 3^x &= 243 \\ \Rightarrow x &= \log_3 243 \\ &= 5. \end{aligned}$$

Note 8.56 See that as $\log_3 x$ is the inverse to the 3^x function, what we are doing here is the same as what we did when we studied manipulation of formulae in Section 5.3. We are "unpacking" to get x by bringing the function to the other side, and in doing so replacing it with its inverse.

Note 8.57 When we ask our calculator, for example, what $\log_3 243$ is, we are asking "3 to the power of **what** equals 243?" When the calculator returns 5, it is saying "3 to the power of **5** equals 243?".

Question 8.58 Find the solution to the following equations.

$$\begin{aligned} 3^x &= 81 \\ 2^x &= 128 \\ 5^x &= 3125 \end{aligned}$$

The logarithm function can also be used when the solution is not an integer.

Question 8.59 Find the solution to the following equations. Do you find the solutions believable?

$$\begin{aligned} 4^x &= 8 \\ 27^x &= 9 \\ 5^x &= 1 \\ 10^x &= 0.001 \\ 2^x &= 5 \end{aligned}$$

Note 8.60 In the final example above, the solution to $2^x = 5$ is approximately $x = 2.3219$. Even though the entire decimal can't be written as a fraction, and we only understand 2^x when x is a fraction, what it is saying is that better fractional approximations to this decimal make 2^x closer to 5. See that

$$\begin{aligned} 2^{2.3} &= 4.92, \\ 2^{2.32} &= 4.993, \\ 2^{2.321} &= 4.9968, \\ 2^{2.319} &= 4.9999. \end{aligned}$$

Each of these decimals can be written as fractions as they terminate, and so can be understood as powers of roots of 2. The idea of 2^x when x is irrational is largely glossed over at a Leaving Cert level, but the curious student can understood 2^x for x irrational as the limit of $2^{\frac{p}{q}}$ for fractions $\frac{p}{q}$ getting closer and closer to x.

Note 8.61 The logarithm function $\log_a x$ is only defined when $a > 0$ and $a \neq 1$, although in practice we will only deal with $\log_a$ for $a = 2, 3, 4, \ldots$ and one other value $e \approx 2.718$ that we will discuss later. It is also only defined for $x > 0$, as the question "a to the power of **what** equals x" only makes sense if $x > 0$, as $a^b > 0$ for all $b \in \mathbb{R}$.

8.8 Logarithm Rules

Much like with indices, there are a number of rules for logarithmic functions. They are given in the middle and right column of page 21 of *Formulae & Tables*.

Séana agus logartaim		Indices and logarithms
$a^p a^q = a^{p+q}$	$\log_a(xy) = \log_a x + \log_a y$	$a^x = y \Leftrightarrow \log_a y = x$
$\frac{a^p}{a^q} = a^{p-q}$	$\log_a\left(\frac{x}{y}\right) = \log_a x - \log_a y$	$\log_a(a^x) = x$
$(a^p)^q = a^{pq}$	$\log_a(x^q) = q\log_a x$	$a^{\log_a x} = x$
$a^0 = 1$	$\log_a 1 = 0$	
$a^{-p} = \frac{1}{a^p}$	$\log_a\left(\frac{1}{x}\right) = -\log_a x$	$\log_b x = \frac{\log_a x}{\log_a b}$
$a^{\frac{1}{q}} = \sqrt[q]{a}$		
$a^{\frac{p}{q}} = \sqrt[q]{a^p} = \left(\sqrt[q]{a}\right)^p$		
$(ab)^p = a^p b^p$		
$\left(\frac{a}{b}\right)^p = \frac{a^p}{b^p}$		

- 21 -

Figure 8.4

Note 8.62 The intuition for the first rule in the middle column is as follows. We know that $\log_2 8 = 3$ and $\log_2 16 = 4$, possibly even without a calculator by knowing that $2^3 = 8$ and $2^4 = 16$. As $8 \times 16 = 128$, if we then ask what $\log_2 128$ it makes sense that the answer is 7, as if $8 = 2^3$ and $16 = 2^4$ then $128 = 2^7$.

More generally, if $\log_a x = b$, $\log_a y = c$, then $x = a^b$, $y = a^c$. Then $xy = a^b a^c = a^{b+c}$, and so the answer to $\log_a(xy)$, i.e. the question "a to the power of what is xy" is "$b+c$".

The second rule has a similar intuition as the first, and the third can be argued for $q \in \mathbb{N}$ by a repeated application of the first (although it is true for all $q \in \mathbb{R}$).

The fourth rule is a consequence of $a^0 = 1$ for all $a \neq 0$. If we ask what $\log_a 1$ is, we are asking "a to the power of **what** equals 1?" The answer is 0.

The fifth rule is a combination of the second and fourth, with x replaced with 1 and y replaced with x.

Note 8.63 The rules on the right column serve to tell us more about $\log_a x$ as a function. In particular, the first rule tells us the definition of the function, and the second and third show us that the $\log_a$ function is the **inverse** of the a^x function (just like $\sqrt[3]{x}$ is the inverse of the x^3 function, etc.).

The final rule on the right column show us how to convert an expression using $\log_b$ into an expression using $\log_a$. It is useful if we know little about $\log_b$ but know more about $\log_a$, or want all logarithms in an equations or expression to be the same.

At a Leaving Cert level, logarithm rules can be applied to solve **logarithmic equations**.

Rule 8.64 To solve logarithmic equations involving constant terms and terms of the form $b\log_a f(x)$ where $b \in \mathbb{Q}$, $a \in \mathbb{N}$ is the same over all logarithmic terms and $f(x)$ is some function of x, we complete the following steps.

1. Write all $b\log_a f(x)$ terms as $\log_a(f(x)^b)$, but if necessary multiply the equation first so that all such $b \in \mathbb{N}$.
2. Use the first and second rules of the middle column to combine terms, until one side of the equals sign is a single logarithmic term and the other side is a constant (possibly 0).
3. Invert the logarithmic function to get an equation without logarithms.
4. Solve the following algebraic expression. At a Leaving Cert level this is likely to be a linear or quadratic equation, although it may initially involve fractions.
5. Check the solutions against the original equation. Some solutions may result in expressions involving the logarithmic of a negative number, which makes the solution invalid.

Note 8.65 The last step is similar to the check we do when solving surd equations (for example $\sqrt{x-2} - \sqrt{x+1} = 1$). At a Leaving Cert level it is likely that we will only use the last step to exclude solutions if we have more than one.

Example 8.66 Solve the equation

$$\log_2(3+x) = 5.$$

We can skip the first two steps. From the definition of the logarithm,

$$\begin{aligned} \log_2(3+x) &= 5 \\ \Rightarrow 3+x &= 2^5 \\ \Rightarrow x &= 29. \end{aligned}$$

This solution is easily seen to be valid.

Example 8.67 Solve the equation

$$\log_3(x) - \log_3(x-1) = 1.$$

We can skip the first step.

$$\begin{aligned} \log_3(x) - \log_3(x-1) &= 1 \\ \Rightarrow \log_3 \frac{x}{x-1} &= 1 \\ \Rightarrow \frac{x}{x-1} &= 3^1 \\ \Rightarrow x &= 3(x-1) \\ &= 3x-3 \\ \Rightarrow 3 &= 2x \\ \Rightarrow \frac{3}{2} &= x. \end{aligned}$$

This solution is valid, as $x-1 > 0$.

Example 8.68 Solve the equation

$$2\log_2(x) - \log_2(x+2) = 0.$$

$$\begin{aligned} 2\log_2(x) - \log_2(x+2) &= 0 \\ \Rightarrow \log_2(x^2) - \log_2(x+2) &= 0 \\ \Rightarrow \log_2 \frac{x^2}{x+2} &= 0 \\ \Rightarrow \frac{x^2}{x+2} &= 2^0 \\ &= 1 \\ \Rightarrow x^2 &= x+2 \\ \Rightarrow x^2 - x - 2 &= 0 \\ \Rightarrow (x-2)(x+1) &= 0 \\ \Rightarrow x &= -1, 2. \end{aligned}$$

See that $x = -1$ results in the first term of the original equation being $2\log_2(-1)$, so that $x = -1$ is invalid and the only solution is $x = 2$.

Question 8.69 Solve the following logarithmic equations.

$$\log_5(x+3) = 2$$
$$\log_4(x-1) = 1$$
$$2\log_7 x = 4$$
$$\log_2(x+5) - \log_2(x-1) = 3$$
$$\log_3(x+1) + \log_3(x-5) = 3$$
$$\log_2(x) - 2\log_2(x-4) = -1$$
$$2\log_3 5 + \log_3 x = 4$$

In problems with logarithms in more than one base, our plan is to convert them so that all logarithms are of the same base, much like we did with indices equations.

Rule 8.70 To solve logarithmic equations involving logarithms with more than one base, convert the logarithms so that they are all of the same base using the rule $\log_b x = \frac{\log_a x}{\log_a b}$. This is best done by re-writing logarithms with large bases in terms of logarithms with small bases. The logarithm can then be solved by applying Rule 8.64 as before.

Example 8.71 Solve the equation

$$\log_3 x - \log_9 x = 1.$$

We first replace the $\log_9$ term with a $\log_3$ term.

$$\begin{aligned}
\log_3 x - \log_9 x &= 1 \\
\Rightarrow \log_3 x - \frac{\log_3 x}{\log_3 9} &= 1 \\
\Rightarrow \log_3 x - \frac{\log_3 x}{2} &= 1 \\
\Rightarrow \frac{1}{2}\log_3 x &= 1 \\
\Rightarrow \log_3 x &= 2 \\
\Rightarrow x &= 3^2 \\
&= 9.
\end{aligned}$$

Example 8.72 Solve the equation

$$\log_2(x+1)-\log_4(x-1)=3.$$

Give your answer to two decimal places.

$$\begin{aligned}
\log_2(x+1)-\log_4(x-1)&=3\\
\Rightarrow \log_2(x+1)-\frac{\log_2(x-1)}{\log_2 4}&=3\\
\Rightarrow \log_2(x+1)-\frac{\log_2(x-1)}{2}&=3\\
\Rightarrow 2\log_2(x+1)-\log_2(x-1)&=6\\
\Rightarrow \log_2((x+1)^2)-\log_2(x-1)&=6\\
\Rightarrow \log_2\frac{(x+1)^2}{x-1}&=6\\
\Rightarrow \frac{(x+1)^2}{x-1}&=2^6\\
\Rightarrow (x+1)^2&=64(x-1)\\
\Rightarrow x^2+2x+1&=64x-64\\
\Rightarrow x^2-62x+65&=0\\
\Rightarrow x&=\frac{62\pm\sqrt{62^2-4(1)(65)}}{2(1)}\\
&=\frac{62\pm\sqrt{3584}}{2}\\
&=1.07, 60.93.
\end{aligned}$$

By checking the original equation, we can see that both solutions are valid.

Question 8.73 Solve the following equations. If necessary give your answer correct to two decimal places.

$$\log_5 x+\log_{125} x=4$$
$$\log_2(x+1)-2\log_{16}(x-1)=1$$
$$\log_3 x-\log_9 x+\log_{27} x-\log_{81} x=1$$

8.9 The Exponential Function and Natural Logarithm

For reasons that will become more clear when we study Calculus, we are often interested in a particular exponent function: e^x where $e \approx 2.718$. It behaves the same as other indices functions, for example

$$\begin{aligned}(e^x)^2 e^4 &= e^{2x} e^4 \\ &= e^{2x+4},\end{aligned}$$

just like if e replaced with 2 or 3. e^x is known as the **exponential function**.

The exponential function e^x, just like other exponent functions, has inverse function $\log_e x$. However we shorten the notation to $\ln x$, which we call the **natural logarithm**. This appears on its own button on your calculator and obeys all the same rules as other logarithms, $\ln xy = \ln x + \ln y$, $\ln 1 = 0$, etc.

The exponential function (and natural logarithm) often come up in real life problems (although again the reason for this will not become clear until we study Calculus).

Example 8.74 The population of a small town t years after January 1st 2020 can be modelled by the function

$$P(t) = 5169 e^{0.02t}.$$

(a) What is the population of the town on January 1st 2020?

(b) What is the population of the town on January 1st 2022? Give your answer correct to the nearest person.

(c) How long after January 1st 2020 will the population of the town reach 10000? Give your answer to the nearest year.

(a) $P(0) = 5169 e^{0.02(0)} = 5169$.

(b) $P(0) = 5169 e^{0.02(2)} \approx 5380$.

(c) We want to know the value of t for which $P(t) = 10000$.

$$\begin{aligned} 10000 &= 5169 e^{0.02t} \\ \Rightarrow \frac{10000}{5169} &= e^{0.02t} \\ \Rightarrow \ln \frac{10000}{5169} &= 0.02t \\ \Rightarrow \frac{1}{0.02} \ln \frac{10000}{5169} &= t \\ \Rightarrow 33 &\approx t. \end{aligned}$$

So the answer is 33 years.

Question 8.75 After being administered a dose of 500 mg (milligrams) of a certain drug, the amount of the drug left in a hospital patient's system t hours after the drug was first administered is given by the function

$$D(t) = 500e^{-0.2t}.$$

(a) How many milligrams of the drug are left in the patient's system after 2 hours?

(b) How long does it take for the amount of the drug in the patient's system to reduce to 200 mg? Give your answer correct to the nearest hour.

Question 8.76 The temperature of a cup of tea left in a room is given by the function

$$T(t) = 20 + 50e^{-0.1t}$$

where t is measured in minutes and $t = 0$ corresponds to the initial time when the cup of tea was made.

(a) What is the initial temperature of the cup of tea?

(b) What is the temperature of the cup of tea after 10 minutes?

(c) How long does it take for the cup of tea to cool down to 30 degrees? Give your answer to the nearest minute.

(d) What is the cup of tea after a long time? What does this tell you about the temperature of the room?

Question 8.77 The population of trout in a lake is in a state of decline. However each year scientists add trout to the lake. They model that the population of trout t years after the scientists start this experiment will be

$$P(t) = 10000 - 5000e^{-0.1t}.$$

(a) What is the initial populatin of the trout?

(b) What is the population of the trout 3 years after the start of the experiment?

(c) Approximately how many years does it take for the trout population to reach 7000?

(d) What is the population of the trout after a long time?

The natural logarithm can also be used to solve problems with other bases. This might be preferred to avoid the awkwardness of using the $\log_{\square}\square$ button on your calculator, or remembering which number goes where.

Note 8.78 To solve an equation in x where x appears in the exponent, for example $2^x = 7$, the natural logarithm of both sides can be taken. The logarithm rule $\ln a^b = b\ln a$ can then be used to "knock the x out of the exponent", making the equation a lot simpler.

Example 8.79 Solve the equation $2^x = 7$ using the natural logarithm. Give your answer correct to two decimal places.

$$\begin{aligned} 2^x &= 7 \\ \Rightarrow \ln 2^x &= \ln 7 \\ \Rightarrow x\ln 2 &= \ln 7 \\ \Rightarrow x &= \frac{\ln 7}{\ln 2} \\ &\approx 2.81. \end{aligned}$$

Note 8.80 There are a couple of things to note in this example. First, while complicated looking and new, $\ln 7$ and $\ln 2$ are just decimals, like $\sqrt{2}$ or $\sin 42°$, which we try to leave as is rather than replacing with decimals. Second, earlier in the chapter we may have instead solved $2^x = 7$ by saying that

$$\begin{aligned} 2^x &= 7 \\ \Rightarrow x &= \log_2 7. \end{aligned}$$

See that

$$\begin{aligned} \log_2 7 &= \frac{\log_e 7}{\log_e 2} \\ &= \frac{\ln 7}{\ln 2} \end{aligned}$$

by our "change of base" logarithm rule; there is no contradiction. However, there is no obligation to use this approach over $\log_2$; it is simply a matter of taste.

Question 8.81 Use the natural logarithm to solve the following equations, giving your answer to two decimal places.

$$\begin{aligned} 3^x &= 8 \\ 5^x &= 2 \\ 2^{3x+1} &= 5 \\ 0.6^x &= 0.1 \end{aligned}$$

8.10 Graphing Indices and Logarithm Functions

Consider the function $f(x) = 2^x$. Before this chapter we understood that

$$f(1) = 2^1 = 2,$$
$$f(2) = 2^2 = 4,$$
$$f(3) = 2^3 = 8,$$

However after studying fractional powers we should also understand, possibly with the help of a calculator, that

$$f\left(\frac{1}{2}\right) = 2^{\frac{1}{2}} = \sqrt{2} = 1.4142\ldots$$
$$f\left(\frac{3}{2}\right) = 2^{\frac{3}{2}} = \sqrt{2^3} = 2.8284\ldots$$

We can do this for all $x \in \mathbb{Q}$. But these numbers don't help us understand the function. To really get a grasp of the function we want to graph it, like we did in Chapter 4.

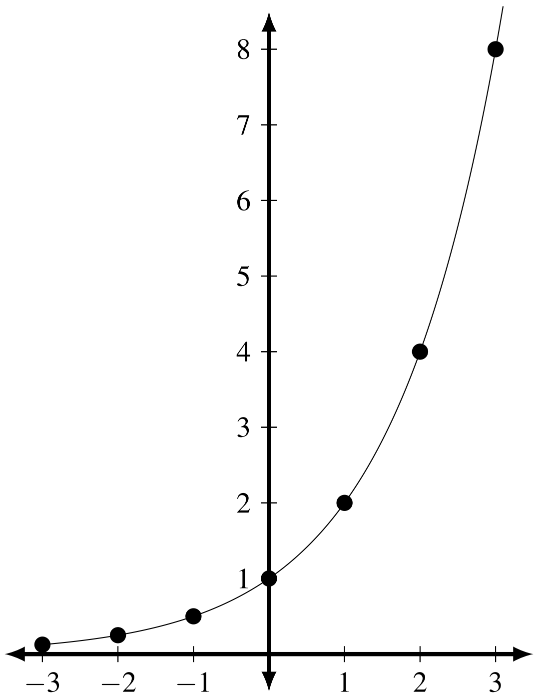


Figure 8.5

As we can see the function is always positive and increasing. The value of the function always doubles as we move one unit to the right, and halves as we move one unit to the left.

From the fact that the function is always increasing, we can get a feel for fractional powers of 2 without knowing what they are. For example, we know that $2^{\frac{13}{4}}$ is some value between $2^3 = 8$ and $2^4 = 16$.

Now that we can visualise the graph of 2^x let's see how it compares to similar graphs. $f(x) = 2^x$ and $g(x) = 3^x$, are shown the same graph below. The axes have been scaled.

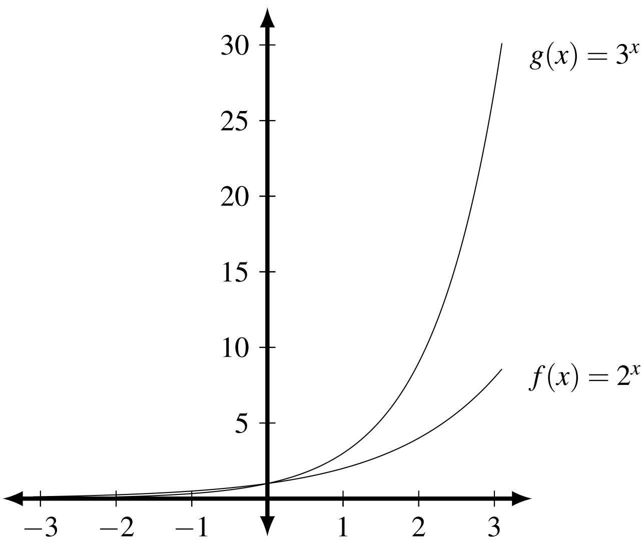


Figure 8.6

While $f(x) = 2^x$ doubles when we move one unit to the right, $g(x) = 3^x$ triples. Similarly, the graph of $g(x)$ is divided by three when we move one unit to the left. As they both agree at $x = 0$, this means that $f(x) > g(x)$ for $x < 0$ as we can see in the graph below that zooms in on the negative x-axis component of the previous graph.

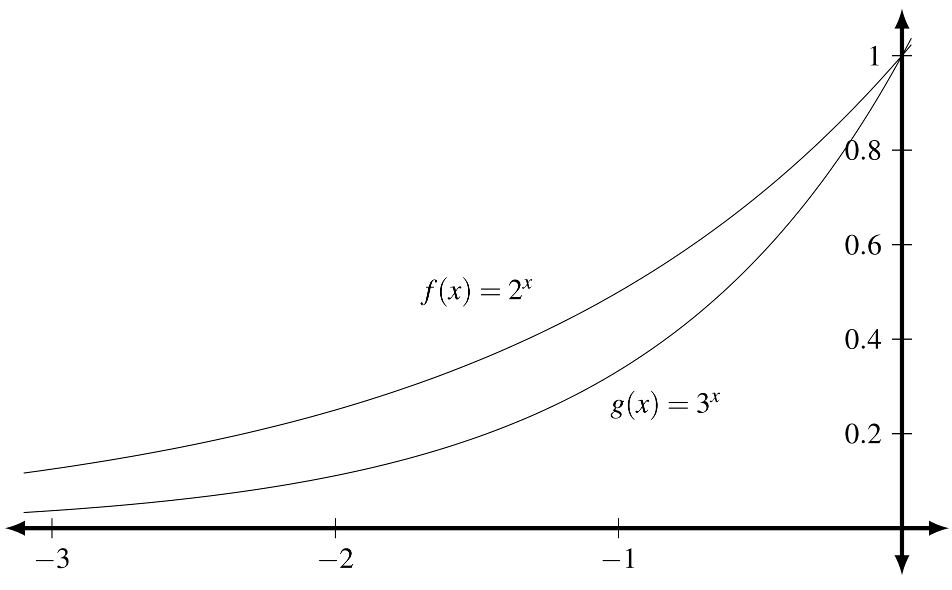


Figure 8.7

Note that both functions get closer and closer to 0 as we move to the left, but **never reach it**.

Graphs of similar functions behave similarly. The graph of the function $h(x) = 5^x$ satisfies $h(0) = 1$ as well, is multiplied by 5 when we move one unit to the right and divides by 5 when we move one unit to the left. Similar statements are true even about functions like $k(x) = 3.4^x$; however things change when we consider a^x for $a < 1$.

Below is the graph of $f(x) = 2^x$ and $l(x) = \left(\frac{1}{2}\right)^x$.

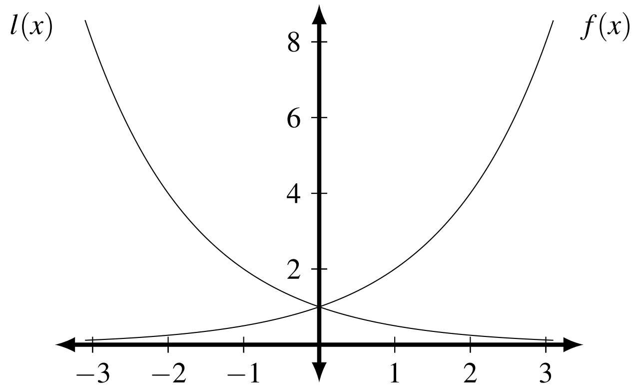


Figure 8.8

Notice that $l(x)$ is multiplied by $\frac{1}{2}$ when we move one unit to the right, and is divided by $\frac{1}{2}$ when we move one unit to the left. In other words, it halves when we move to the right and doubles when we move to the left; the opposite of the behaviour of $f(x)$. In fact, the graph of $l(x)$ can be seen as the graph of $f(x)$ "flipped across the y-axis". Also for $l(x)$, the functions gets closer and closer to 0 as we move to the right, i.e. when x is a large positive number, rather than the left. This makes sense if we realise that we can write

$$\begin{aligned} l(x) &= \left(\frac{1}{2}\right)^x \\ &= \frac{1}{2^x} \\ &= 2^{-x} \\ &= f(-x). \end{aligned}$$

Remembering our work on manipulating graphs in Section 4.4, when x is replaced with $-x$ the graph flips across the y-axis, so this shouldn't be surprising.

Question 8.82 The graph of $f(x) = 3^x$ is drawn on the graph below. Roughly draw the graph of $g(x) = 4^x$, $h(x) = 2^x$ and $k(x) = \left(\frac{1}{3}\right)^x$ on the same graph.

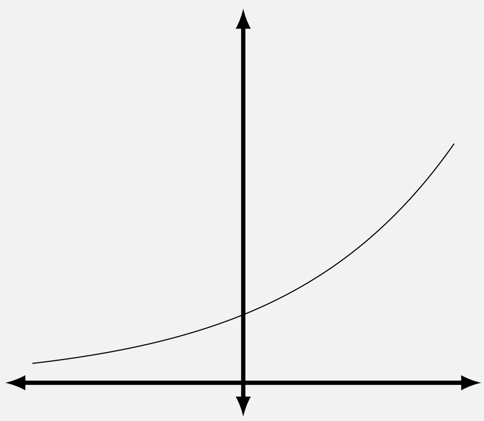


Figure 8.9

Regarding logarithm functions, remember that logarithms are the inverse of indices functions. Below is the graph of 2^x and $\log_2 x$, along with the dashed $y = x$ line usually drawn to help us draw inverses.

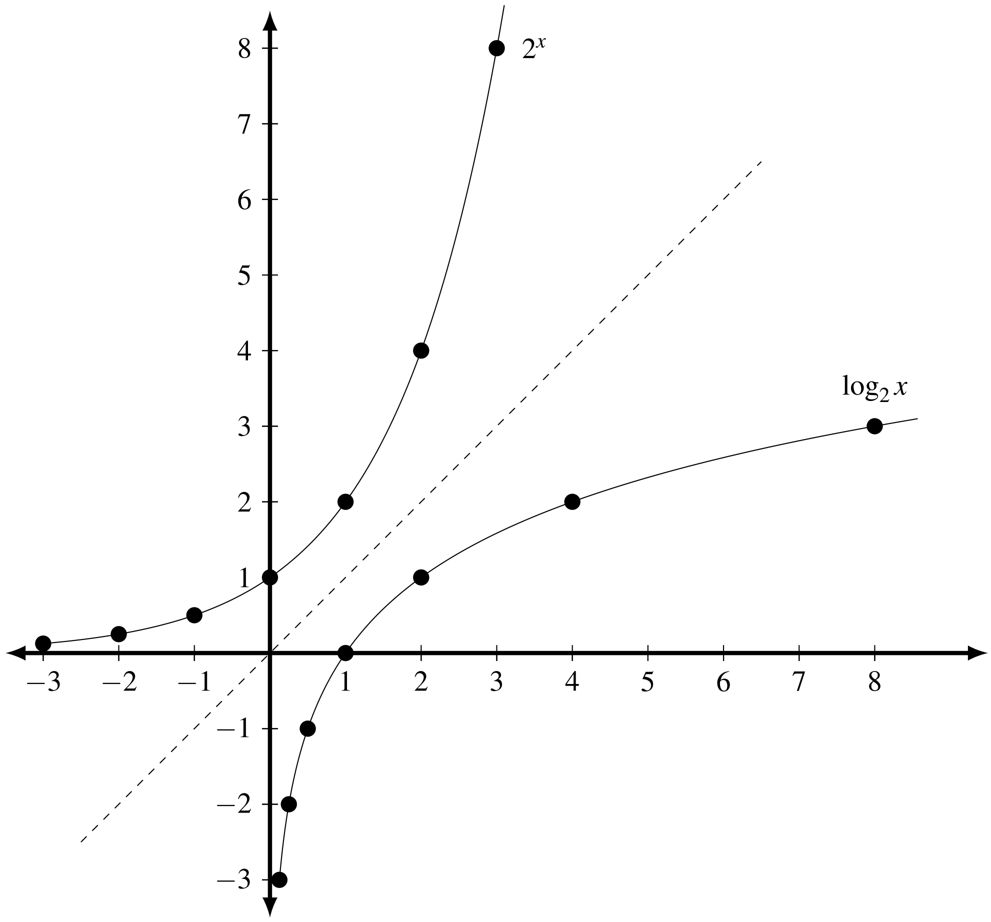


Figure 8.10

Note 8.83 Much like how all indices functions are positive (i.e. their range is $(0,\infty)$), logarithm functions are only defined for $x > 0$ (i.e. their domain is $(0,\infty)$). Similarly, much like how indices functions have domain $\mathbb{R}$, logarithm functions have range $\mathbb{R}$, i.e. they are subjective.

All indices functions pass through the point $(0,1)$. Similarly all logarithm functions pass through the point $(1,0)$.

Finally, indices functions a^x are decreasing for $a < 1$. The corresponding logarithm function $\log_a x$ is also decreasing rather than increasing. These logarithm functions, however, are rarely studied.

For the sake of completeness, the graphs of 0.5^x and $\log_0.5x$ are shown below, although as stated before in this chapter $\log_a$ for $a < 1$ is rarely used at Leaving Cert level.

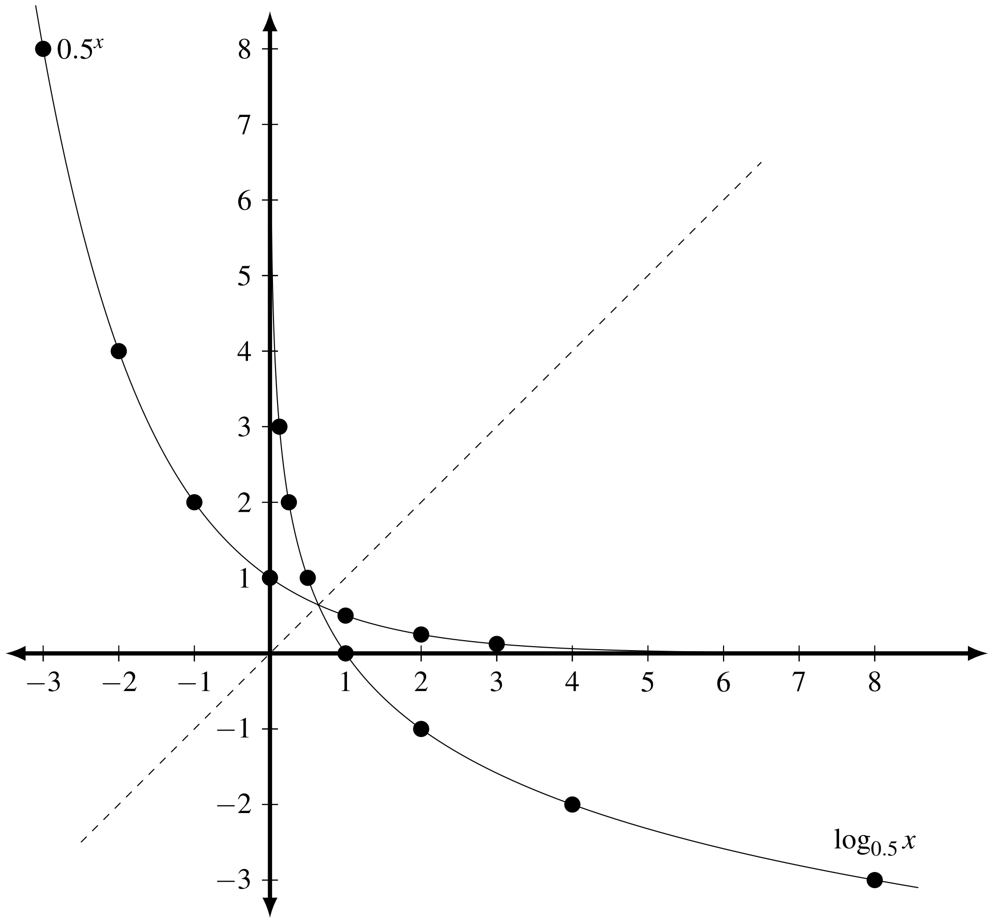


Figure 8.11

8.11 Exponentials and Logarithms in Inequalities

Some problems, especially those derived from practical problems, result in inequalities involving indices terms.

Rule 8.84 To solve problems involving indices and inequalities, they can be solved using the **natural logarithm** as if they were equations, with one caveat. If dividing by $\ln a$ for $a < 1$, $\ln a < 0$ and so the sign must change direction. It is recommended to use the natural logarithm rather than logarithms with particular bases here to avoid awkward questions like is $\log_{0.6} 2 > 0$?

Example 8.85 Solve the inequality

$$5(3^x) - 10 < 110.$$

First we isolate for the 3^x, then take logarithms, then solve for x. No flipping of the sign is required.

$$\begin{aligned} 5(3^x) - 10 &< 110 \\ \Rightarrow 5(3^x) &< 120 \\ \Rightarrow 3^x &< 24 \\ \Rightarrow \ln 3^x &< \ln 24 \\ \Rightarrow x \ln 3 &< \ln 24 \\ \Rightarrow x &< \frac{\ln 24}{\ln 3} \\ \Rightarrow x &< 2.89. \end{aligned}$$

Example 8.86 Solve the inequality

$$(0.8)^x < 0.1.$$

Note that when we divide by $\ln 0.8$, we flip the sign because $\ln 0.8 < 0$.

$$\begin{aligned} 0.8^x &< 0.1 \\ \Rightarrow \ln 0.8^x &< \ln 0.1 \\ \Rightarrow x \ln 0.8 &< \ln 0.1 \\ \Rightarrow x &> \frac{\ln 0.1}{\ln 0.8} \\ \Rightarrow x &> 10.32. \end{aligned}$$

Question 8.87 Solve the following inequalities.

$$3000(1.05)^x \leq 5000$$
$$50(0.4)^x < 10$$
$$10 - 5(0.6)^x > 7$$
$$50 - 20e^{-2t} > 40$$

Question 8.88 After being administered a dose of 500 mg (milligrams) of a certain drug, the amount of the drug left in a hospital patient's system t hours after the drug was first administered is given by the function

$$D(t) = 500e^{-0.2t}.$$

(a) When does the patient have more than 300 mg of the drug in their system?

(b) When does the patient have less than 200 mg of the drug in their system?

Question 8.89 The temperature of a cup of tea left in a room is given by the function

$$T(t) = 20 + 50e^{-0.1t}$$

where t is measured in minutes and $t = 0$ corresponds to the initial time when the cup of tea was made.

(a) When is the temperature of the cup of tea greater than 40 degrees?

(b) When is the temperature of the cup of tea less than 30 degrees?

(c) When is the temperature of the cup of tea less than 20 degrees?

8.12 Hidden Quadratics

Now that we can solve quadratics, for example, equations such as

$$x^2 - 13x + 36 = 0,$$

what if x is replaced with a more complicated expression? Take, for example, the equation

$$x^4 - 13x^2 + 36 = 0.$$

As $x^4 = (x^2)^2$, we can view this as a quadratic in x^2. Maybe more simply, by letting $y = x^2$, $y^2 = x^4$ and this equation becomes

$$y^2 - 13y + 36 = 0.$$

This is a quadratic that we can solve as normal.

$$\begin{aligned} y^2 - 13y + 36 &= 0 \\ \Rightarrow (y-4)(y-9) &= 0 \\ \Rightarrow y &= 4, 9. \end{aligned}$$

However y is just a variable we made up; we wanted to solve the equation

$$x^4 - 13x^2 + 36 = 0,$$

i.e. we wanted to find x. However as $y = x^2$,

$$\begin{aligned} y &= 4, 9 \\ \Rightarrow x^2 &= 4, 9 \\ \Rightarrow x &= \pm 2, \pm 3. \end{aligned}$$

Therefore although this is technically a quartic equation, it can be reduced to a quadratic equation by letting $y = x^2$, solved for y and then solved for x. This is an example of a "hidden quadratic".

Example 8.90 Solve the equation

$$x^4 - 17x^2 + 16 = 0.$$

Letting $y = x^2$, $y^2 = x^4$ and

$$\begin{aligned} x^4 - 17x^2 + 16 &= 0 \\ \Rightarrow y^2 - 17y + 16 &= 0 \\ \Rightarrow (y-16)(y-1) &= 0 \\ \Rightarrow y &= 1, 16 \\ \Rightarrow x^2 &= 1, 16 \\ \Rightarrow x &= \pm 1, \pm 4. \end{aligned}$$

This can be applied to a wider variety of problems than ones where we let $y = x^2$.

Example 8.91 Solve the equation

$$x^6 + 7x^3 - 8 = 0.$$

Letting $y = x^3$, $y^2 = x^6$ and

$$\begin{aligned} x^6 + 7x^3 - 8 &= 0 \\ \Rightarrow y^2 + 7y - 8 &= 0 \\ \Rightarrow (y-1)(y+8) &= 0 \\ \Rightarrow y &= 1, -8 \\ \Rightarrow x^3 &= 1, -8 \\ \Rightarrow x &= \sqrt[3]{1}, \sqrt[3]{-8} \\ &= 1, -2. \end{aligned}$$

Sometimes even though we get a solution for y it doesn't result in any solutions for x.

Example 8.92 Solve the equation

$$x^4 - 2x^2 - 8 = 0.$$

Letting $y = x^2$, $y^2 = x^4$ and

$$\begin{aligned} x^4 - 2x^2 - 8 &= 0 \\ \Rightarrow y^2 - 2y - 8 &= 0 \\ \Rightarrow (y-4)(y+2) &= 0 \\ \Rightarrow y &= -2, 4 \\ \Rightarrow x^2 &= -2, 4. \end{aligned}$$

$x^2 = -2$ gives no solutions, but

$$\begin{aligned} x^2 &= 4 \\ \Rightarrow x &= \pm 2, \end{aligned}$$

so our solutions are $x = \pm 2$.

Question 8.93 Solve the following equations.

$$\begin{aligned} x^4 - 10x^2 + 9 &= 0 \\ x^4 - 25x^2 + 144 &= 0 \\ x^6 + 26x^3 - 27 &= 0 \\ x^4 - 8x^2 - 9 &= 0 \\ x^4 + 6x^2 + 8 &= 0 \end{aligned}$$

This approach can also be applied to equations involving terms such as 2^x. For example, given the equation

$$2^{2x} - 6(2^x) + 8 = 0,$$

if we realise that $2^{2x} = (2^x)^2$, then by letting $y = 2^x$ we have $y^2 = 2^{2x}$. Then

$$\begin{aligned} 2^{2x} - 6(2^x) + 8 &= 0 \\ \Rightarrow y^2 - 6y + 8 &= 0 \\ \Rightarrow (y-4)(y-2) &= 0 \\ \Rightarrow y &= 2, 4. \end{aligned}$$

Again, we don't care about y per se, we care about x. Now that we know y,

$$\begin{aligned} y &= 2, 4 \\ \Rightarrow 2^x &= 2 \text{ or } 2^x = 4 \\ \Rightarrow 2^x &= 2^1 \text{ or } 2^x = 2^2 \\ \Rightarrow x &= 1 \text{ or } x = 2. \end{aligned}$$

In other words, once we solve for 2^x we find x by solving a simple version of the problems we studied in Section 8.6, or by taking logarithms.

Example 8.94 Solve the following equation.

$$3^{2x} - 12(3^x) + 27 = 0.$$

Letting $y = 3^x$, $y^2 = 3^{2x}$ and

$$\begin{aligned} 3^{2x} - 12(3^x) + 27 &= 0 \\ \Rightarrow y^2 - 12y + 27 &= 0 \\ \Rightarrow (y-3)(y-9) &= 0 \\ \Rightarrow y &= 3, 9 \\ \Rightarrow 3^x &= 3 \text{ or } 3^x = 9 \\ \Rightarrow \ln 3^x &= \ln 3 \quad \text{or} \quad \ln 3^x = \ln 9 \\ \Rightarrow x \ln 3 &= \ln 3 \quad \text{or} \quad x \ln 3 = \ln 9 \\ \Rightarrow x &= 1 \quad \text{or} \quad x = \frac{\ln 9}{\ln 3} \\ \Rightarrow x &= 1, 2. \end{aligned}$$

Since we're doing the same thing to both solutions for y, we don't need to split up the solutions into separate equations.

Example 8.95 Solve the following equation.

$$4(2^{2x}) - 33(2^x) + 8 = 0.$$

Letting $y = 2^x$, $y^2 = 2^{2x}$ so that

$$\begin{aligned} 4y^2 - 33y + 8 &= 0 \\ \Rightarrow 4y^2 - 32y - y + 8 &= 0 \\ \Rightarrow 4y(y-8) - (y-8) &= 0 \\ \Rightarrow (4y-1)(y-8) &= 0 \\ \Rightarrow 4y - 1 = 0 &\text{ or } y - 8 = 0 \\ \Rightarrow y &= \frac{1}{4}, 8 \\ \Rightarrow 2^x &= \frac{1}{4}, 8 \\ \Rightarrow \ln 2^x &= \ln\frac{1}{4}, \ln 8 \\ \Rightarrow x\ln 2 &= \ln\frac{1}{4}, \ln 8 \\ \Rightarrow x &= \frac{\ln\frac{1}{4}}{\ln 2}, \frac{\ln 8}{\ln 2} \\ &= -2, 3. \end{aligned}$$

Because we can never have $a^x < 0$ for any $a > 0$, we may again have a solution for y that doesn't result in a solution for x. We may also get decimal solutions.

Example 8.96 Solve the following equation.

$$3^{2x} - 24(3^x) - 25 = 0.$$

Letting $y = 3^x$, $y^2 = 3^{2x}$ and

$$\begin{aligned} 3^{2x} - 24(3^x) - 25 &= 0 \\ \Rightarrow y^2 - 24y - 25 &= 0 \\ \Rightarrow (y+1)(y-25) &= 0 \\ \Rightarrow y &= -1, 25 \\ \Rightarrow 3^x &= -1, 25 \\ \Rightarrow \ln 3^x &= \ln(-1), \ln 25 \\ \Rightarrow x\ln 3 &= \ln 25 \quad (\text{as } \ln(-1) \text{ undefined}) \\ \Rightarrow x &= \frac{\ln 25}{\ln 3} \\ &= 2.93. \end{aligned}$$

Finally, it is possible that powers need to be "peeled away" in order to make these equations look like hidden quadratics.

Example 8.97 Solve the following equation.

$$3^{2x} - 4(3^{x+1}) + 27 = 0.$$

First we "peel away" one power of 3 in 3^{x+1}.

$$\begin{aligned} 3^{2x} - 4(3^{x+1}) + 27 &= 0 \\ \Rightarrow 3^{2x} - 4(3^1 3^x) + 27 &= 0 \\ \Rightarrow 3^{2x} - 12(3^x) + 27 &= 0. \end{aligned}$$

This is now the same equation as in Example 8.94; we can proceed as in that example to get $x = 1, 2$.

Finally, sometimes powers need to be "shifted" to make them look like hidden quadratics.

Example 8.98 Solve the equation

$$e^x - 2 = e^{-x}.$$

We first multiply each side by e^x.

$$\begin{aligned} e^x - 2 &= e^{-x} \\ \Rightarrow e^x e^x - 2e^x &= e^{-x} e^x \\ \Rightarrow e^{2x} - 2e^x &= 1. \end{aligned}$$

We then let $y = e^x$ so that $y^2 = e^{2x}$ and

$$\begin{aligned} e^{2x} - 2e^x &= 1 \\ \Rightarrow y^2 - 2y &= 1 \\ \Rightarrow y^2 - 2y - 1 &= 0 \\ \Rightarrow (y-2)(y+1) &= 0 \\ \Rightarrow y &= 2, -1 \\ \Rightarrow e^x &= 2, -1 \\ \Rightarrow x &= \ln 2, \ln(-1) \\ \Rightarrow x &= \ln 2 \quad \text{(as } \ln(-1) \text{ is undefined).} \end{aligned}$$

Question 8.99 Solve the following equations.

$$2^{2x} - 3(2^x) + 2 = 0$$
$$3^{2x} - 28(3^x) + 27 = 0$$
$$5^{2x+1} - 126(5^x) + 25 = 0$$
$$7^{2x} - 5(7^x) - 14 = 0$$
$$2^{2x} + 5(2^x) + 6 = 0$$
$$e^{2x} - 4(e^x) + 3 = 0$$
$$2^{x-1} + 2^{-x+1} - 2 = 0$$

8.13 A Return to the Binomial Theorem

In Section 2.5 we learned about the Binomial Theorem (page 20 of *Formulae & Tables*).

An Teoirim dhéthéarmach **Binomial theorem**

$$(x+y)^n = \sum_{r=0}^{n} \binom{n}{r} x^{n-r} y^r = \binom{n}{0} x^n + \binom{n}{1} x^{n-1} y + \binom{n}{2} x^{n-2} y^2 + \cdots + \binom{n}{r} x^{n-r} y^r + \cdots + \binom{n}{n} y^n$$

comhéifeachtaí déthéarmacha $\binom{n}{r} = {}^nC_r = C(n,r) = \frac{n!}{r!(n-r)!}$ binomial coefficients

Figure 8.12

We used this to expand powers of linear functions of x such as $(3x+4)^5$. We will now consider more varied powers of x.

Example 8.100 Fully expand $(4x^2+2)^4$.

We replace x and y with $4x^2$ and 2 in the formula. We take the entirety of each term to the required power, remembering our rules of indices.

$$\begin{aligned}
&(4x^2+2)^4 \\
=&\binom{4}{0}(4x^2)^4 + \binom{4}{1}(4x^2)^3(2) + \binom{4}{2}(4x^2)^2(2)^2 + \binom{4}{3}(4x^2)(2)^3 + \binom{4}{4}2^4 \\
=&256x^8 + (4)(64x^6)(2) + 6(16x^4)(4) + 4(4x^2)(8) + 16 \\
=&256x^8 + 512x^6 + 384x^4 + 128x^2 + 16.
\end{aligned}$$

Note 8.101 Just like x^2 is a power of x, $\frac{1}{x} = x^{-1}$ is also a power of x, which is why we study them here.

Example 8.102 Fully expand $\left(3x - \frac{2}{x}\right)^3$.

$$\begin{aligned}
&\left(3x - \frac{2}{x}\right)^3 \\
=&\binom{3}{0}(3x)^3 + \binom{3}{1}(3x)^2\left(-\frac{2}{x}\right) + \binom{3}{2}(3x)\left(-\frac{2}{x}\right)^2 + \binom{3}{3}\left(-\frac{2}{x}\right)^3 \\
=&27x^3 + 3(9x^2)\left(-\frac{2}{x}\right) + 3(3x)\left(\frac{4}{x^2}\right) + \left(-\frac{8}{x^3}\right) \\
=&27x^3 - 54x + \frac{36x}{x^2} - \frac{8}{x^3} \\
=&27x^3 - 54x + \frac{36}{x} - \frac{8}{x^3}.
\end{aligned}$$

Note 8.103 Notice how in both Examples 8.100 and 8.102 the powers of x formed an arithmetic (linear) sequence; 8, 6, 4, 2, 0 and 3, 1, -1, -3. This will always be the case when expanding a binomial of the form $(ax^p + bx^q)^n$ for $a, b \in \mathbb{R}$, $p, q \in \mathbb{Z}$.

Question 8.104 Fully expand the following expressions.

$$(x^2+5)^4$$

$$\left(2x + \frac{1}{x}\right)^4$$

$$\left(3 - \frac{2}{x}\right)^3$$

$$\left(2x - \frac{1}{x}\right)^4$$

$$\left(2x^2 + \frac{1}{x}\right)^5$$

If we are asked to find a specific term in a binomial expansion where the term in the brackets is not linear, it can be harder to know which term we are looking for. This can be done either by guesswork, or more analytically.

Example 8.105 Calculate the x^4 term in the expansion of $(4x^2-2)^9$.

We know that all terms in the binomial expansion look like

$$\binom{9}{}(4x^2)^{\square}(-2)^{\square}$$

We could guess the numbers, but using a more analytical approach, let the power of $4x^2$ be r:

$$\binom{9}{}(4x^2)^{r}(-2)^{\square}.$$

Then the power of -2 must be $9-r$, and the lower number in $\binom{9}{}$ can be either r or $9-r$:

$$\binom{9}{r}(4x^2)^{r}(-2)^{9-r}.$$

Let's ignore the constants and focus only on x. We want the value of r that gives us x^r. See that

$$\binom{9}{r}(4x^2)^{r}(-2)^{9-r}=Cx^{2r}$$

for some constant C. So we need

$$\begin{aligned}2r&=4\\ \Rightarrow r&=2\end{aligned}$$

so that our term is

$$\begin{aligned}\binom{9}{2}(4x^2)^{2}(-2)^{7}&=36(16x^4)(-128)\\ &=-73,728x^4.\end{aligned}$$

Note 8.106 In the previous example, there is nothing wrong with guessing the term to be

$$\binom{9}{2}(4x^2)^{2}(-2)^{7}$$

straight away. Just make sure that the term you end up with does indeed include an x^4 term.

Example 8.107 Calculate the x^3 term in the expansion of $(3x+\frac{1}{x})^7$.

We know that, for some r, our term is of the form

$$\begin{aligned}\binom{7}{r}(3x)^r\left(\frac{1}{x}\right)^{7-r} &= Cx^r\left(\frac{1}{x^{7-r}}\right)\\ &= C\frac{x^r}{x^{7-r}}\\ &= Cx^{r-(7-r)}\\ &= Cx^{2r-7}\end{aligned}$$

for some constant C. We need

$$\begin{aligned}2r-7&=3\\ \Rightarrow r&=5\end{aligned}$$

so that our term is

$$\begin{aligned}\binom{7}{5}(3x)^5\left(\frac{1}{x}\right)^2 &= 21(243x^5)\left(\frac{1}{x^2}\right)\\ &= 5,103x^3.\end{aligned}$$

Example 8.108 Calculate the constant term in the expansion of $(2x^2-\frac{3}{x})^9$.

We know that, for some r, our term is of the form

$$\begin{aligned}\binom{9}{r}(2x^2)^r\left(-\frac{3}{x}\right)^{9-r} &= Cx^{2r}\left(\frac{1}{x^{9-r}}\right)\\ &= C\frac{x^{2r}}{x^{9-r}}\\ &= Cx^{2r-(9-r)}\\ &= Cx^{3r-9}\end{aligned}$$

for some constant C. For this term to be constant we need

$$\begin{aligned}3r-9&=0\\ \Rightarrow r&=3\end{aligned}$$

so that our term is

$$\begin{aligned}\binom{9}{3}(2x^2)^3\left(-\frac{3}{x}\right)^6 &= (56)8x^6\frac{729}{x^6}\\ &= 489,888.\end{aligned}$$

Question 8.109 Calculate the x^6 term in the following binomial expansions.

$$(3+2x^2)^8$$
$$(7x-x^2)^6$$
$$\left(\frac{2}{x}-x\right)^8$$
$$\left(\frac{2}{x}+x^2\right)^6$$

8.14 Summary

Some parts of this chapter are foundational tools relevant to general algebra, for example squaring fractions and taking square roots of powers. Exponentials and logarithms, natural and otherwise, can be present in abstract algebra problems. These problems can be related to expressions, equations, inequalities or functions. They are also present in more practical problems.

Further to being present in standalone exam questions, the natural logarithm and e^x are a significant presence in Calculus (Chapters 15 and 16), and are foundational to some parts of Financial Maths (Chapter 19) and Sequences and Series (Chapter 17). However we will not return to study indices and logarithms (or binomial expansions) at a higher level or in more significant detail; we have learned all we need to know.

8.15 Homework

Integer Rules of Indices

1. Write the following expressions as x^n for some $n \in \mathbb{N}$.

$$x^6x^3$$
$$x^2x^4$$
$$x^4x^4$$
$$x^7x^2x$$

2. Write the following expressions as x^n for some $n \in \mathbb{N}$.

$$\left(x^2\right)^3$$
$$\left(x^4\right)^5$$
$$\left(x^3\right)^3$$
$$x^4\left(x^2\right)^5$$
$$\left(x^4x^3\right)^6$$

3. Write the following expressions as x^n for some $n \in \mathbb{N}$.

$$\frac{x^6}{x^2}$$
$$\frac{x^7}{x^5}$$
$$\frac{x^{10}}{x^3}$$
$$\frac{x^4}{x}$$
$$\frac{x^6}{x^3x}$$
$$\frac{x^2x^7}{x^3}$$

4. Write the following expressions as x^n for some $n \in \mathbb{Z}$

$$\frac{x^2}{x^6}$$
$$\frac{x^4}{x^7}$$
$$\frac{x^2}{x^5}$$
$$\frac{x}{x^5}$$

5. Write the following as $\frac{1}{x^n}$ for some $n \in \mathbb{N}$.

$$x^{-2}$$
$$x^{-5}$$
$$x^{-8}$$

6. Write the following without indices.

$$2^{-1}$$
$$3^{-4}$$
$$4^{-2}$$
$$5^{-3}$$
$$6^0$$

Fractional Rules of Indices

7. Write the following in the form $\sqrt[n]{x^m}$ or $\sqrt[n]{x}^m$ for some $m, n \in \mathbb{N}$.

$$x^{\frac{2}{3}}$$
$$x^{\frac{4}{7}}$$
$$x^{\frac{1}{3}}$$

8. Calculate the following without a calculator.

$$16^{\frac{1}{2}}$$
$$27^{\frac{1}{3}}$$
$$8^{\frac{2}{3}}$$
$$9^{\frac{3}{2}}$$

Combining Indices with Multiplication/Division

9. Get rid of the brackets in the following expressions.

$$(4x)^2$$
$$(5y)^3$$
$$(2z^2)^4$$
$$\left(\frac{x}{4}\right)^2$$
$$\left(\frac{3}{y}\right)^4$$
$$\left(\frac{x^2}{y}\right)^5$$
$$\left(\frac{x}{5y^2}\right)^2$$

10. Write the following expressions without a root.

$$\sqrt{9x^2}$$
$$\sqrt[3]{8y^9}$$
$$\sqrt{\frac{x^2}{y^6}}$$
$$\sqrt{\frac{25}{16}}$$
$$\sqrt[3]{\frac{64}{27}}$$
$$\sqrt[3]{\frac{x^{12}}{27}}.$$

Simplifying Expressions Involving Indices

11. Write the following expressions as x^n for some $n \in \mathbb{Q}$.

    (a) $\frac{x^2x^5}{x^3}$

    (b) $\frac{x^3\sqrt{x}}{x}$

    (c) $\sqrt[3]{x^5x^4}$

12. Write the expression

$$\frac{27\sqrt{3}}{3^x}$$

as 3^{ax+b} for some $a, b \in \mathbb{Q}$.

13. Write the expression

$$\frac{25^x}{125}$$

as 5^{ax+b} for some $a, b \in \mathbb{Q}$.

14. Write the expression

$$\left(\frac{8\sqrt{2}}{2^x}\right)^2$$

as 2^{ax+b} for some $a, b \in \mathbb{Q}$.

15. Write the expression

$$27\sqrt{9^x}$$

as p^{ax+b} for some $a, b \in \mathbb{Q}$, $p \in \mathbb{N}$.

16. Write the expression

$$\frac{8^x}{4}$$

as p^{ax+b} for some $a, b \in \mathbb{Q}$, $p \in \mathbb{N}$.

17. Write the expression

$$\frac{25^x\sqrt{125}}{125^x}$$

as p^{ax+b} for some $a, b \in \mathbb{Q}$, $p \in \mathbb{N}$.

Solving Equations Involving Indices

18. Solve the following equations.

    (a) $3\sqrt{27} = 3^x$

    (b) $\frac{5}{\sqrt[3]{25^x}} = 125$

    (c) $4\sqrt{2}\,(2^x)^2 = \sqrt{32}$

Logarithms

19. Calculate the following logarithms without a calculator.

    (a) $\log_2 4$

    (b) $\log_3 27$

    (c) $\log_{10} 10,000$

    (d) $\log_7 1$

    (e) $\log_5 \frac{1}{5}$

    (f) $\log_2 \frac{1}{4}$

Logarithm Rules

20. Solve the following equations. If necessary give your answer(s) to two decimal places.

(a) $\log_2(x+1)=3$

(b) $\log_3(x-1)-\log_3(2x+1)=-1$

(c) $\log_3(x+2)+\log_3(2x+1)=2$

(d) $\log_2(x-1)+\log_2(3x-7)=5$

(e) $2\log_5 x-\log_5(x+5)=-1$

(f) $\log_7(x-1)+\log_7(x^2+x+1)=1$

(g) $2\log_3(x-1)-\log_3(x^2-1)=2$ (Hint: You may want to factorise the fraction)

(h) $\log_2(x-1)+2\log_4(x+1)=3$

21. (a) Show that $\log_a b=\frac{1}{\log_b a}$

(b) Hence solve the equation

$$\frac{1}{\log_2 x}+\frac{1}{\log_3 x}+\frac{1}{\log_6 x}=2.$$

22. It is given that

$$x^3y^4=e^2,$$
$$x^4y^8=1.$$

(a) Let $p=\ln x$, $q=\ln y$. Use the above equations to show that

$$3p+4q=2,$$
$$4p+8q=0.$$

(b) Solve the simultaneous equations found in part (a) to find p,q.

(c) Hence find x, y.

The Exponential Function and Natural Logarithm

23. The amount of money in an investment account t years after it was initially invested is given by the function $M(t)=3000(1.05)^t$.

(a) How much money was initially invested?

(b) How much money is in the account after 5 years?

(c) How long does it take for the amount of money in the investment account to reach €12,000?

(d) When is there more than €15000 in the account?

24. An object falls out of a plane. Its speed t seconds after falling out of the plane (in km/hr) is given by the function $v(t)=30-30e^{-0.8t}$.

(a) What is its speed 1 second after falling out of the plane?

(b) How long does it take for the object to reach a speed of 25 km/hr?

(c) When is the object's speed less than 28 km/hr?

(d) If the object falls for a long time, what will its speed be?

Hidden Quadratics

25. Solve the following equations.

(a) $x^4-104x^2+400=0$

(b) $x^4-68x^2+256=0$

(c) $x^6-19x^3-216=0$

(d) $4x^4-37x^2+9=0$

(e) $x^4-22x^2-75=0$

26. Solve the following equations.

(a) $2^{2x}-17(2^x)+16=0$

(b) $5^{2x}-30(5^x)+125=0$

(c) $3(3^{2x})-28(3^x)+9=0$

(d) $4^{2x}-4^x-12=0$

A Return to the Binomial Theorem

27. Fully expand the following expressions.

    (a) $(3-4x^2)^4$

    (b) $\left(\frac{2}{x}+1\right)^3$

    (c) $\left(\frac{2}{x^2}-3x\right)^4$

    (d) $\left(x-\frac{3}{x}\right)^5$

    (e) $\left(2x^2+\frac{1}{x}\right)^4$

28. For each of the following expressions, find the x^n term where n is given in the question.

    (a) $(5+2x^2)^9$ — x^2

    (b) $(3x^2-4x)^8$ — x^{11}

    (c) $\left(1-\frac{3}{x}\right)^9$ — $\frac{1}{x^7}$

    (d) $\left(\frac{2}{x}-3x\right)^6$ — constant term

    (e) $\left(\frac{3}{x}-2x\right)^6$ — x^2

    (f) $\left(x^2+\frac{4}{x}\right)^9$ — x^6

8.16 Homework Solutions

Integer Rules of Indices

1.

$$x^9$$
$$x^6$$
$$x^8$$
$$x^{10}$$

2.

$$x^6$$
$$x^{20}$$
$$x^9$$
$$x^{14}$$
$$x^{42}$$

3.

$$x^4$$
$$x^2$$
$$x^7$$
$$x^3$$
$$x^2$$
$$x^6$$

4.

$$x^{-4}$$
$$x^{-3}$$
$$x^{-3}$$
$$x^{-4}$$

5.

$$\frac{1}{x^2}$$
$$\frac{1}{x^5}$$
$$\frac{1}{x^8}$$

6.

$$\frac{1}{2}$$
$$\frac{1}{81}$$
$$\frac{1}{16}$$
$$\frac{1}{125}$$
$$1$$

Fractional Rules of Indices

7.

$$\sqrt[3]{x^2}$$
$$\sqrt[7]{x^4}$$
$$\sqrt[3]{x}$$

8.

$$4$$
$$3$$
$$4$$
$$27$$

Combining Indices with Multiplication/Division

9.

$$16x^2$$

$$125y^3$$

$$16z^8$$

$$\frac{x^2}{16}$$

$$\frac{81}{y^4}$$

$$\frac{x^{10}}{y^5}$$

$$\frac{x^2}{25y^4}$$

10.

$$3x$$

$$2y^3$$

$$\frac{x}{y^3}$$

$$\frac{5}{4}$$

$$\frac{4}{3}$$

$$\frac{x^4}{3}$$

Simplifying Expressions Involving Indices

11.

$$x^4$$

$$x^{\frac{5}{2}}$$

$$x^3$$

12.

$$3^{\frac{7}{2}-x}$$

13.

$$5^{2x-3}$$

14.

$$2^{7-2x}$$

15.

$$3^{3+x}$$

16.

$$2^{3x-2}$$

17.

$$5^{\frac{3}{2}-x}$$

Solving Equations Involving Indices

18. (a) $x = \frac{5}{2}$
(b) $x = -3$
(c) $x = 0$

Logarithms

19. (a) 2
(b) 3
(c) 4
(d) 0
(e) −1
(f) −2

Logarithm Rules

20. (a) $x = 7$
(b) $x = 4$
(c) $x = 1$
(d) $x = 5$
(e) $x \approx 2.19$
(f) $x = 2$
(g) $x = \frac{5}{4}$

21. (a) $\log_a b = \frac{\log_b b}{\log_b a} = \frac{1}{\log_b a}$

(b) $x = 6$

22. It is given that

$$x^3y^4 = e^2,$$
$$x^4y^8 = 1.$$

(a)

$$\begin{aligned} x^3y^4 &= e^2 \\ \Rightarrow \ln(x^3y^4) &= \ln e^2 \\ \Rightarrow \ln(x^3)+\ln(y^4) &= 2\ln e \\ \Rightarrow 3\ln x + 4\ln y &= 2 \\ \Rightarrow 3p+4q &= 2 \end{aligned}$$

(b) $p = 2, q = -1$

(c) $x = e^2, y = e^{-1}$

The Exponential Function and Natural Logarithm

23. (a) €3000

(b) €3828.84

(c) 28.41 years

(d) $t > 32.99$ years

24. (a) 16.52 km/hr

(b) 2.23 seconds

(c) $t < 3.39$ seconds

(d) It will be very close to 30 km/hr, but it will never quite get there

Hidden Quadratics

25. (a) $x = \pm 2, \pm 10$

(b) $x = \pm 2, \pm 8$

(c) $x = -2, 3$

(d) $x = \pm\frac{1}{2}, \pm 3$

(e) $x = \pm 5$

26. (a) $x = 0, 4$

(b) $x = 1, 2$

(c) $x = -1, 2$

(d) $x = 1$

A Return to the Binomial Theorem

27. (a) $256x^8 - 768x^6 + 864x^4 - 432x^2 + 81$

(b) $\frac{8}{x^3} + \frac{12}{x^2} + \frac{6}{x} + 1$

(c) $\frac{16}{x^8} - \frac{96}{x^5} + \frac{216}{x^2} - 216x + 81x^4$

(d) $x^5 - 15x^3 + 90x - \frac{270}{x} + \frac{405}{x^3} - \frac{243}{x^5}$

(e) $16x^8 + 32x^5 + 24x^2 + \frac{8}{x} + \frac{1}{x^4}$

28. (a) $7,031,250x^2$

(b) $-1,548,288x^{11}$

(c) $-\frac{78,732}{x^7}$

(d) $-4,320$

(e) $2,160x^2$

(f) $32,256x^6$

8.17 Harder Problems

The problems in this section are designed to be harder than homework problems, but are not designed to be Leaving Cert problems. They're arguably not even helpful in revising for the Leaving Cert exam, and are more for advanced students to test their general understanding of the work in this section at a high level.

1. What is your best argument that $2^x > 2^y$ if $x > y$? Assume $x, y \in \mathbb{Q}$.
2. What is your best argument that, if $2^x = 2^y$, then we have to have $x = y$? Assume we know that $x, y \in \mathbb{Q}$.
3. When we apply ln to inequalities, we don't change the sign. For example,

$$\begin{aligned} 2^x &> 7 \\ \Rightarrow \ln 2^x &> \ln 7. \end{aligned}$$

   Why is this a valid operation?
4. Can you prove the five logarithm rules given in the middle column of page 21 of *Formulae & Tables*? You can assume the indices rules on the left column.
5. In logarithm equations, we sometimes had an extra solution which we discarded because it made the expression inside the logarithm negative. Can you explain how such values end up as solutions to our final algebraic equation, either in one particular example like Example 8.68 or in general?

8.18 Harder Problems Solutions

1. Say $x > y$. See that, by dividing by 2^y which is positive,

$$\begin{aligned} 2^x &> 2^y \\ \Leftrightarrow 2^{x-y} &> 0. \end{aligned}$$

So we really only need to show that $2^z > 1$ if $z > 0$. Assume $z = \frac{p}{q}$ for some $\frac{p}{q} > 0$. We can assume then that $p, q > 0$. Then

$$\begin{aligned} 2^z &= 2^{\frac{p}{q}} \\ &= \sqrt[q]{2^p}. \end{aligned}$$

It's clear that $2^p > 1$, so we only need to show that the q-th root of a number greater than 1 is greater than 1. This is fairly quick. Assume by contradiction that there is some $0 \leq r \leq 1$ such that

$$\sqrt[q]{2^p} = r.$$

Then

$$2^p = r^q.$$

But if $0 \leq r \leq 1$ then $0 \leq r^q \leq 1$, meaning $2^p \leq 1$ which is nonsense. By contradiction $r > 1$ and so $2^z > 1$, as required.

2. Say that $2^x = 2^y$ for some x, y. Then

$$2^{x-y} = 1.$$

Let $x - y = \frac{p}{q}$ for some $p, q \in \mathbb{Z}$. Then

$$\begin{aligned} 2^{\frac{p}{q}} &= 1 \\ \Rightarrow \sqrt[q]{2^p} &= 1 \\ \Rightarrow 2^p &= 1^q \\ &= 1. \end{aligned}$$

This is impossible for any $p > 0$ in $\mathbb{Z}$ or $p < 0$ in $\mathbb{Z}$. Therefore $p = 0$ and so $x = y$.

3. It is because $\ln x$ is an **increasing** function. When applying an increasing function to an inequality the inequality is preserved. See Question 7.12 of the Harder Problems in Chapter 7 for more discussion on this.

4. First let's prove that

$$\log_a(xy) = \log_a x + \log_a y.$$

Let

$$\begin{aligned} p &= \log_a x, \\ q &= \log_a y, \\ r &= \log_a(xy). \end{aligned}$$

Then

$$\begin{aligned} a^p &= x, \\ a^q &= y, \\ a^r &= xy \\ &= a^p a^q \\ &= a^{p+q}. \end{aligned}$$

So

$$\begin{aligned} r &= p + q \\ \Rightarrow \log_a(xy) &= \log_a x + \log_a y \end{aligned}$$

as required.

Now let's prove that

$$\log_a\left(\frac{x}{y}\right) = \log_a x - \log_a y.$$

Let

$$\begin{aligned} p &= \log_a x, \\ q &= \log_a y, \\ r &= \log_a\left(\frac{x}{y}\right). \end{aligned}$$

Then

$$\begin{aligned} a^p &= x, \\ a^q &= y, \\ a^r &= \frac{x}{y} \\ &= \frac{a^p}{a^q} \\ &= a^{p-q}. \end{aligned}$$

So

$$\begin{aligned} r &= p - q \\ \Rightarrow \log_a\left(\frac{x}{y}\right) &= \log_a x - \log_a y \end{aligned}$$

as required.

Now let's prove that

$$\log_a(x^q) = q\log_a x.$$

Let

$$\begin{aligned} p &= \log_a x, \\ r &= \log_a(x^q). \end{aligned}$$

Then

$$\begin{aligned} a^p &= x, \\ a^r &= x^q \\ &= (a^p)^q \\ &= a^{pq}. \end{aligned}$$

So

$$\begin{aligned} r &= pq \\ \Rightarrow \log_a(x^q) &= q\log_a x \end{aligned}$$

as required.

The proof that $\log_a 1 = 0$ is true by the definition of the $\log_a$ function; $\log_a 1$ is asking "a to the power of **what** gives 1?" The answer is 0.

The proof that

$$\log_a\left(\frac{1}{x}\right) = -\log_a x$$

is immediate by letting $q = -1$ in the third rule.

5. Often these values make the expression inside another logarithm function negative in a way that cancels out when we combine the logarithms, either because two negative expressions get multiplied or one gets squared. For example, in Example 8.68, let's just see what the equation looks like at each step of the workings when we input $x = -1$ (the false solution) at the start.

$$\begin{aligned} 2\log_2(-1) - \log_2(1) &= 0 \\ \Rightarrow \log_2(-1)^2 - \log_2(1) &= 0 \end{aligned}$$

which is true. Looking at the analogous operations for general x,

$$\begin{aligned} 2\log_2(x) - \log_2(x+2) &= 0 \\ \Rightarrow \log_2(x^2) - \log_2(x+2) &= 0. \end{aligned}$$

See that, from the second line onwards (although we don't show the rest of the calculations here) $x = -1$ is a solution of the equation. However it is not a solution to the equation in the first line.

Something similar happens to the equation

$$\log_2(x+1)+\log_2(x-1)=3.$$

Solving this equation as we normally would in this chapter gives the solutions $x=3,-3$ and we discard $x=-3$. $x=-3$ is not a solution to the first line of the workings because it makes both expresssions inside the logarithm terms negative, but from the second line in our workings onwards $x=-3$ is a solution:

$$\log_2((x+1)(x-1))=3.$$

This is because the negative signs cancel out here and there is nothing invalid about $x=-3$ in this equation.

8.19 Revision

Rules of Indices

1. Write the following expressions as x^a for some $a \in \mathbb{Q}$.

$$x^3x^6$$

$$x^5x$$

$$\frac{x^6}{x^2}$$

$$\frac{x^3}{x^7}$$

$$\frac{x^2x^3}{x^4}$$

$$x^{-2}x^5$$

$$\frac{1}{x^4}$$

$$\sqrt{x}$$

$$\sqrt[3]{x^2}$$

$$\sqrt[5]{x^7}$$

$$\frac{1}{\sqrt{x}}$$

Solving Equations Involving Indices

2. Solve the following equations.

(a) $5^2 5^x = 5^7$

(b) $\frac{3^x}{27} = 81$

(c) $\frac{\sqrt{2}}{(2^x)^3} = 8$

(d) $\frac{\sqrt[3]{4^x}}{16} = 4^x$

(e) $\frac{4^x}{8} = 16^2$

Logarithm Rules

3. Solve the following equations. If necessary give your answer(s) to two decimal places.

(a) $\log_2(5x-2) = 3$

(b) $\log_5(3x+13) - \log_5(x+1) = 1$

(c) $\log_3(2x+1) + \log_3(x-1) = 3$

(d) $\log_2(x) - 2\log_{16}(x+1) = 2$

(e) $\ln x + \ln(x+1) - \ln(x+2) = 0$

Hidden Quadratics

4. Solve the following equations.

(a) $x^4 - 10x^2 + 9 = 0$

(b) $x^4 - x^2 - 12 = 0$

(c) $4x^4 - 65x^2 + 16 = 0$

(d) $x^6 - 63x^3 - 64 = 0$

(e) $4^{2x} - 20(4^x) + 64 = 0$

(f) $5^{2x} - 6(5^x) + 5 = 0$

(g) $3^{2x} - 7(3^x) - 18 = 0$

(h) $8(2^{2x}) - 6(2^x) + 1 = 0$

A Return to the Binomial Theorem

5. Fully expand the following expression using the Binomial Theorem.

(a) $(x+x^2)^5$

(b) $(4x^2+1)^4$

(c) $\left(\frac{3}{x}+2\right)^3$

(d) $\left(\frac{4}{x^2}-2\right)^5$

(e) $\left(2x-\frac{5}{x^2}\right)^4$

6. In the following expressions, find the given term if the expansion were to be expanded out.

   (a) $(x^2+1)^9$ x^4

   (b) $(3x^2+2x)^8$ x^10

   (c) $\left(2x^2-\frac{3}{x}\right)^7$ x^2

   (d) $\left(5x-\frac{1}{x}\right)^6$ x^2

   (e) $\left(\frac{3}{x}+2x^2\right)^9$ constant term

8.20 Revision Solutions

Rules of Indices

1.

x^9

x^6

x^4

x^{-4}

x^1

x^3

x^{-4}

$x^{\frac{1}{2}}$

$x^{\frac{2}{3}}$

$x^{\frac{7}{5}}$

$x^{-\frac{1}{2}}$

Solving Equations Involving Indices

2. (a) $x = 5$
   (b) $x = 7$
   (c) $x = -\frac{5}{6}$
   (d) $x = -3$
   (e) $x = \frac{11}{2}$

Logarithm Rules

3. (a) $x = 2$
   (b) $x = 4$
   (c) $x = 4$
   (d) $x = 8 + 4\sqrt{5}$
   (e) $x = \sqrt{2}$

Hidden Quadratics

4. (a) $x = \pm 1, \pm 3$
   (b) $x = \pm 2$
   (c) $x = \pm\frac{1}{2}, \pm 4$
   (d) $x = 4, -1$
   (e) $x = 1, 2$
   (f) $x = 0, 1$
   (g) $x = 2$
   (h) $x = -1, -2$

A Return to the Binomial Theorem

5. (a) $x^5 + 5x^6 + 10x^7 + 10x^8 + 5x^9 + x^{10}$
   (b) $256x^8 + 256x^6 + 96x^4 + 16x^2 + 1$
   (c) $\frac{27}{x^3} + \frac{54}{x^2} + \frac{36}{x} + 8$
   (d) $\frac{1024}{x^{10}} - \frac{2560}{x^8} + \frac{2560}{x^6} - \frac{1280}{x^4} + \frac{320}{x^2} - 32$
   (e) $\frac{625}{x^8} - \frac{1000}{x^5} + \frac{600}{x^2} - 160x + 16x^4$

6. (a) $36x^4$
   (b) $(3x^2 + 2x)^8 \quad x^10$
   (c) $16,128x^2$
   (d) $9,375x^2$
   (e) $489,888$

VI Shape & Measure

9 Geometry . . . 431
9.1 Angles & Parallel Lines . . . 431
9.2 Triangles . . . 434
9.3 Quadrilaterals & Parallelograms . . . 441
9.4 Circles . . . 444
9.5 Proofs . . . 449
9.6 General Geometry Problems . . . 456
9.7 Summary . . . 463
9.8 Homework . . . 464
9.9 Homework Solutions . . . 469
9.10 Harder Problems . . . 470
9.11 Harder Problems Solutions . . . 471
9.12 Revision . . . 472
9.13 Revision Solutions . . . 478

10 Constructions . . . 479
10.1 Notes on the Syllabus . . . 479
10.2 Junior Cert Constructions . . . 480
10.3 Leaving Cert Constructions . . . 494
10.4 Summary . . . 507

11 Trigonometry . . . 509
11.1 Right Angled Triangles . . . 509
11.2 Special Angles and the Range of the Trigonometric Functions . . . 516
11.3 The Unit Circle . . . 520
11.4 Trigonometric Equations 1 . . . 525
11.5 The Sine and Cosine Rule . . . 527
11.6 Choosing Between the Sine and Cosine Rule . . . 540
11.7 Problems Involving Isosceles Triangles . . . 541
11.8 The Area of a Triangle . . . 542
11.9 Problems With Multiple Triangles . . . 546
11.10 Three Dimensional Problems . . . 551
11.11 Angles outside of 0° to 360° . . . 559
11.12 Trigonometric Equations 2 . . . 560
11.13 Radians . . . 563
11.14 The Graphs of the Trigonometric Functions . . . 570
11.15 Modelling with Trigonometric Functions . . . 575
11.16 Formulae 1-24, and $\sec$, $\csc$ and $\cot$ . . . 586
11.17 Application of Trigonometric Formulae . . . 589
11.18 Required Proofs . . . 596
11.19 Summary . . . 602
11.20 Homework . . . 603
11.21 Homework Solutions . . . 618
11.22 Harder Problems . . . 622
11.23 Harder Problems Solutions . . . 623
11.24 Revision . . . 625
11.25 Revision Solutions . . . 638

12 Area, Volume & Surface Area . . . 639
12.1 Formulae . . . 639
12.2 Applying the Formulae . . . 642
12.3 Combining Shapes, and Harder Problems . . . 656
12.4 Enlargements . . . 658
12.5 Summary . . . 660
12.6 Homework . . . 661
12.7 Homework Solutions . . . 664
12.8 Harder Problems . . . 665
12.9 Revision . . . 666

9. Geometry

9.1 Angles & Parallel Lines

Given a diagram showing angles and lines, some of which may be parallel, we have the following facts that help us to calculate angles.

Theorem 9.1

1. Any angles on a straight line add to 180°.

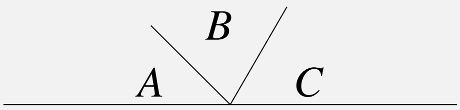


Figure 9.1: $A+B+C=180^\circ$

2. Vertically opposite angles: angles opposite each other at the intersection of two lines, are equal.

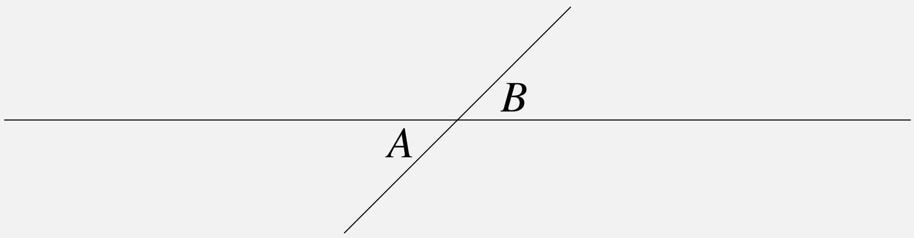


Figure 9.2: $A=B$

3. Corresponding angles: angles on the same side of a line that cuts two lines are equal if and only if the lines are parallel.

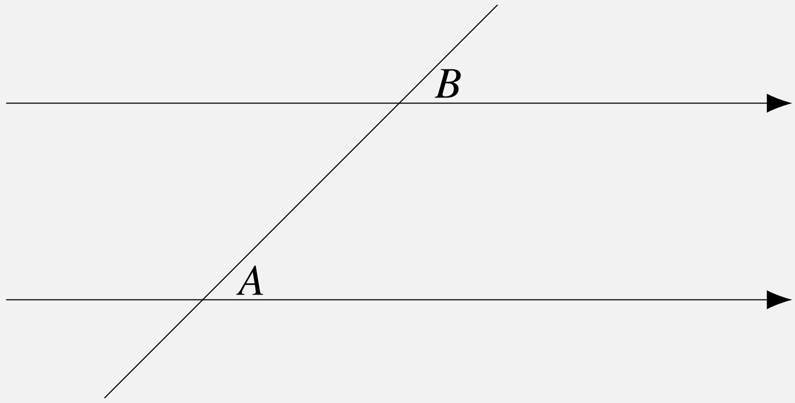


Figure 9.3: $A = B$

4. Alternate angles: angles on the opposite side of a line that cuts two lines are equal if and only if the lines are parallel.

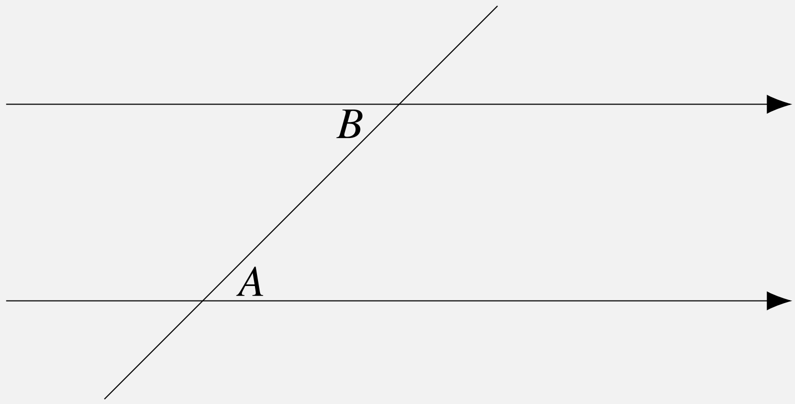


Figure 9.4: $A = B$

5. Interior angles: angles that share one line add to 180° if and only if the other two lines that make the angles are parallel.

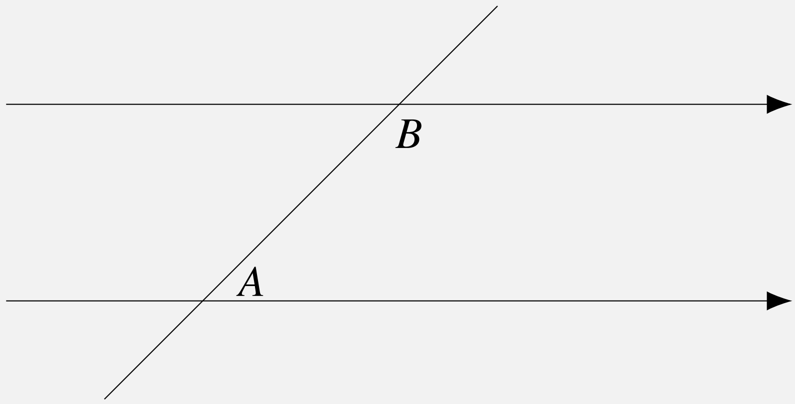


Figure 9.5: $A + B = 180^\circ$

Example 9.2 Calculate the value of the angles A, B, C.

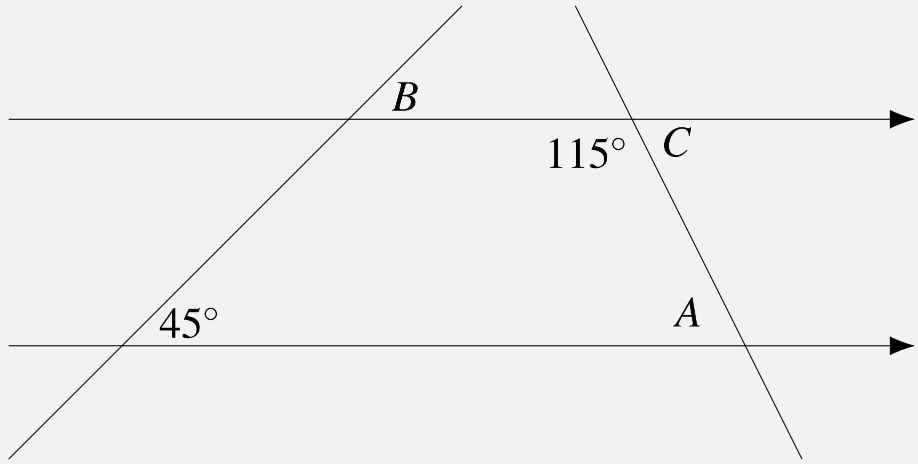


Figure 9.6

$$B = 45^\circ \text{ (corresponding angles)}$$
$$A + 115^\circ = 180^\circ \text{ (interior angles)}$$
$$\Rightarrow A = 65^\circ$$
$$C = A \text{ (alternate angles)}$$
$$= 65^\circ.$$

Note 9.3 State why each relationship is true (alternate angles, interior angels, etc.) when answering questions in this chapter. This is in line with Leaving Cert marking schemes.

Note 9.4 Parallel lines in these notes will usually be drawn with arrows like the above. In the case of two sets of parallel lines two arrows may be shown for one set.

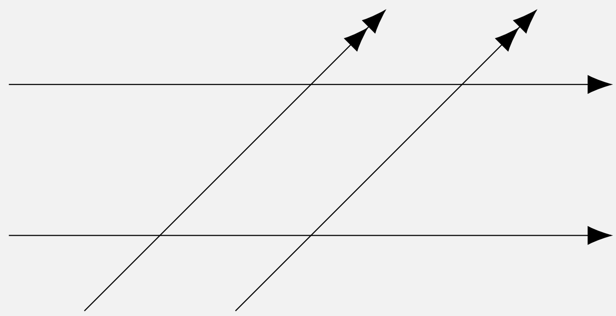


Figure 9.7

In exams students will not be expected to memorise notation as it is not universally standardised. Instead the quesion will say if lines are parallel.

Question 9.5 Calculate the value of the angles A, B, C, D, E, F.

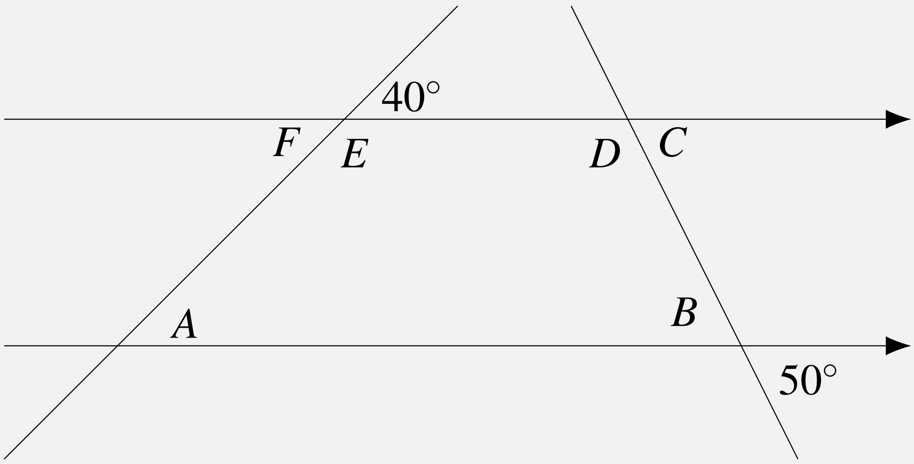


Figure 9.8

9.2 Triangles

Theorem 9.6

1. The sum of the angles of a triangle add up to 180°.
2. The exterior angle of a triangle is equal to the sum of the two interior opposite angles.

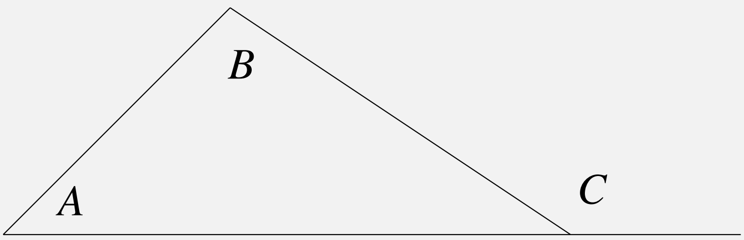


Figure 9.9: $A + B = C$

3. Two sides of a triangle are equal if and only if the opposite angles are equal. These triangles are called isosceles triangles.
4. Three sides of a triangle are equal if and only if all three angles are equal, i.e. equal to 60°. These triangles are called equilateral triangles.

Note 9.7 As another point of notation, when we say AB we mean a line or line segment. However $|AB|$ represents the length of the line segment from A to B and so is a number. The angle $\angle ABC$ is the angle made by the pair of line segments AB and BC. The angle is always at the position of the middle point (in this case B). Usually the angle as a number also has vertical bars $|\angle ABC|$ but the bars are sometimes dropped in these notes.

Note 9.8 In these notes sides in a triangle that are equal are shown with a line crossed through them. If more than one collection of sides are equal, two lines may be used for the second collection, as in the following diagram. In this case, $|AD| = |CD|$, but they are not necessarily equal to $|AC|$ and $|AB|$.

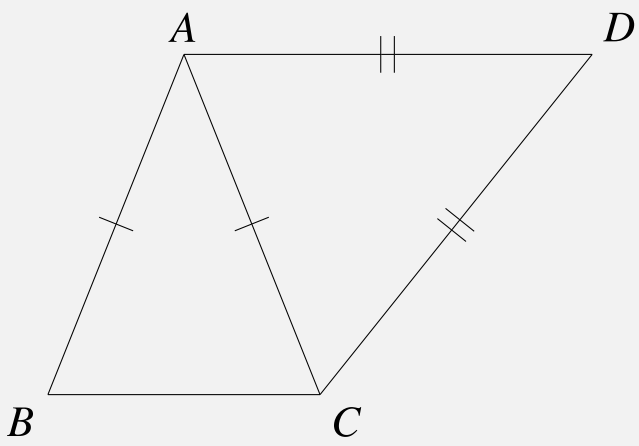


Figure 9.10

Less commonly similar notation will be used to show that angles are equal.

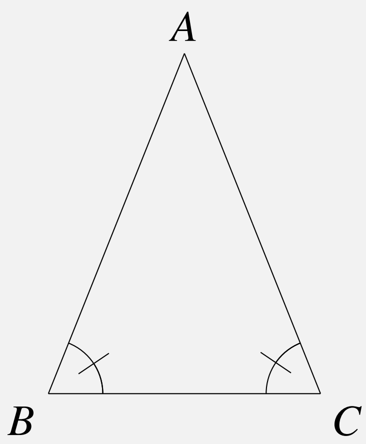


Figure 9.11

Like with parallel lines, in an exam students will also be told lines are equal in length, rather than being expected to memorise non-standard notation.

Note 9.9 Regarding isosceles triangles as mentioned in Theorem 9.6, consider the triangle below. The lines AB and AC are equal in length if and only if the angles $\angle ABC$ and the angles $\angle ACB$ are equal, as they are the angles **opposite** those lines. Therefore knowing one pair of lines or angles are equal, we know immediately that the other pair are as well.

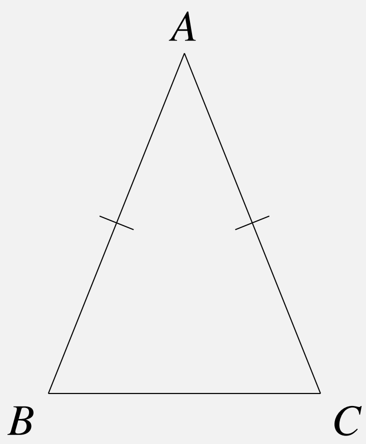


Figure 9.12

We can combine Theorem 9.6 with our work in Section 9.1 to calculate the length of angles.

Example 9.10 Calculate the values of the angles A, B, C, D.

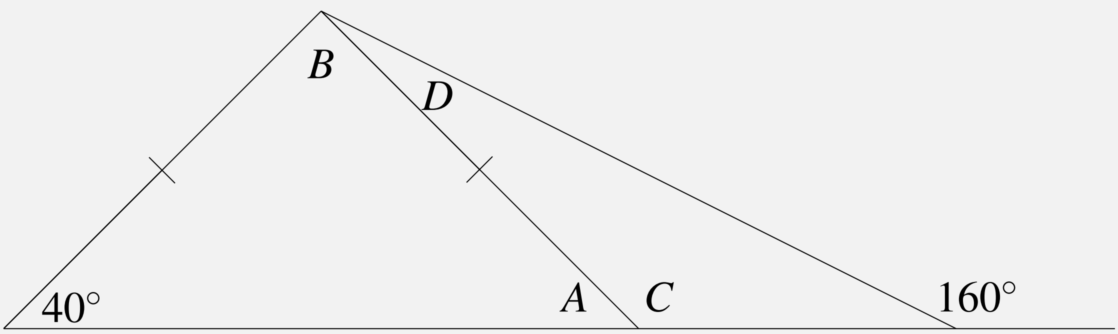


Figure 9.13

$$\begin{aligned}
A &= 40^\circ \text{ (isosceles triangle)} \\
40^\circ + A + B &= 180^\circ \text{ (triangle)} \\
\Rightarrow B &= 100^\circ \\
A + C &= 180^\circ \text{ (straight line)} \\
\Rightarrow C &= 140^\circ \\
C + D &= 160^\circ \text{ (exterior angle)} \\
\Rightarrow D &= 20^\circ.
\end{aligned}$$

Question 9.11 Calculate the angles A, B, C.

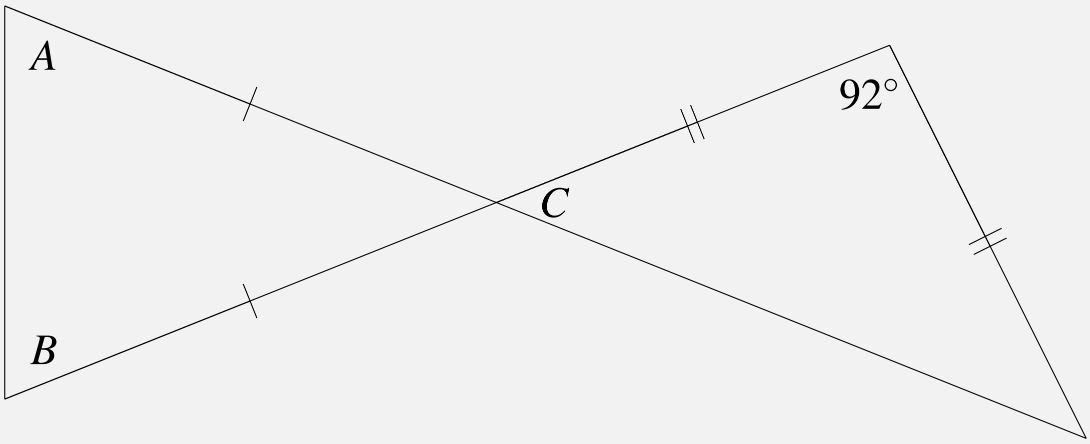


Figure 9.14

Question 9.12 Calculate the angles A, B, C.

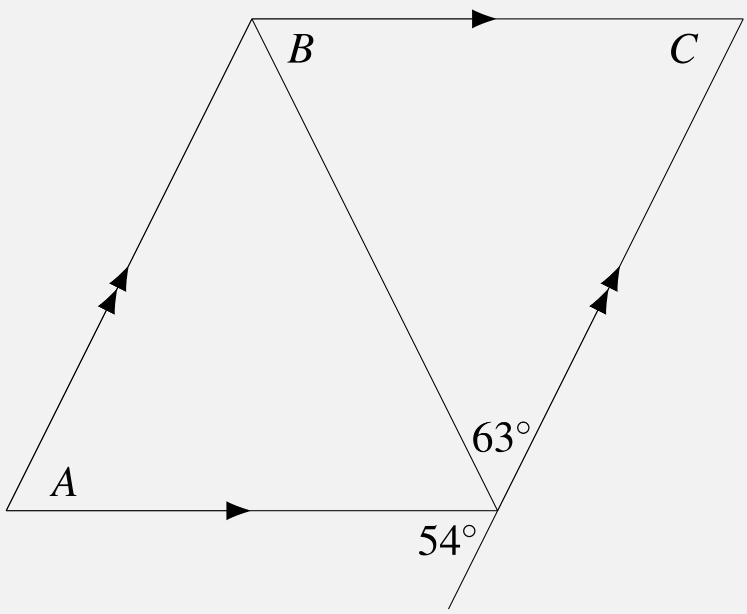


Figure 9.15

In addition to Theorem 9.6 we have another property of triangles to help us solve problems, in particular the idea of identical, or congruent triangles.

Definition 9.13 Two triangles are **congruent** if they are identical up to rotation and reflection. More formally, they are congruent if they have exactly the same three sides and angles.

Theorem 9.14 For two triangles to be congruent it is enough that they have one of the following collections of the same three measurements, consecutive as you rotate around the triangle.

1. Side, Angle, Side (SAS)

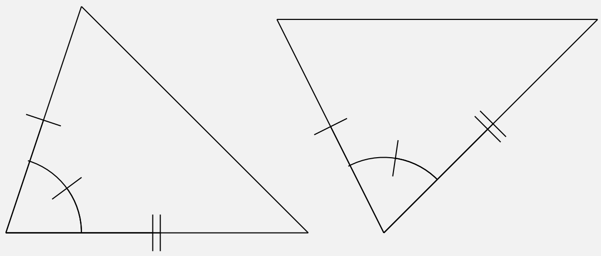


Figure 9.16

2. Angle, Side, Angle (ASA)

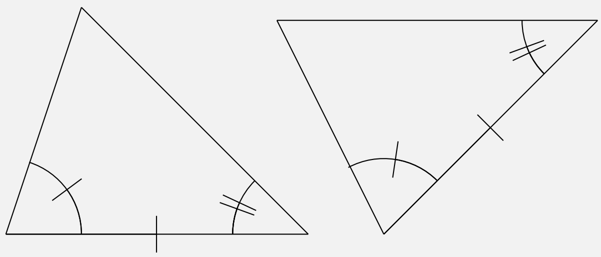


Figure 9.17

3. Side, Side, Side (SSS)

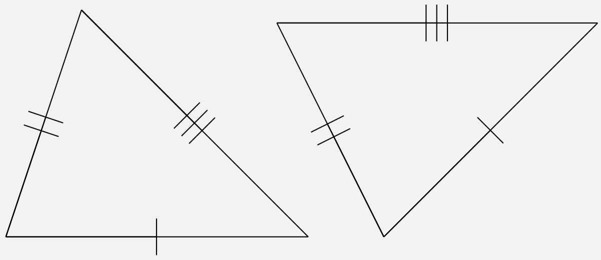


Figure 9.18

4. Right angle, Side, Hypotenuse (RSH)

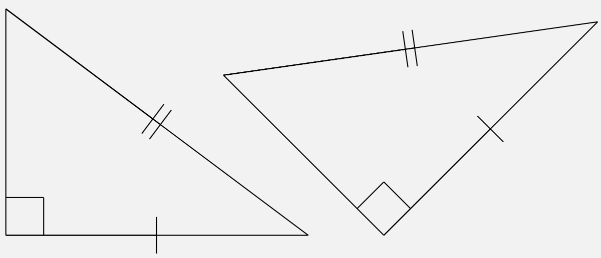


Figure 9.19

Example 9.15 In the diagram below, show that the triangles pqr and psr are congruent.

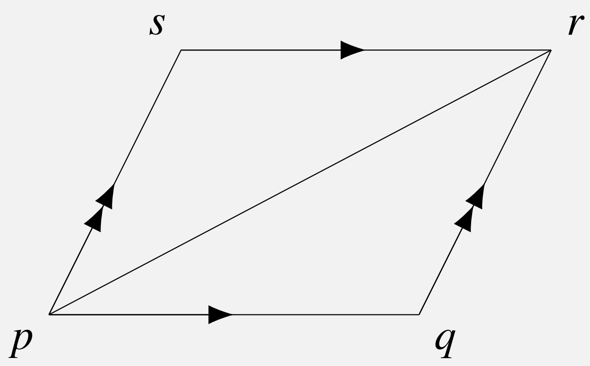


Figure 9.20

$$|\angle spr| = |\angle prq| \quad \text{(alternate angles)}$$
$$|\angle rpq| = |\angle srp| \quad \text{(alternate angles)}$$
$$|pr| = |pr|$$

so that $\triangle pqr$ and $\triangle psr$ are congruent (ASA).

Question 9.16 px is perpendicular to qr. Show that it also bisects qr.

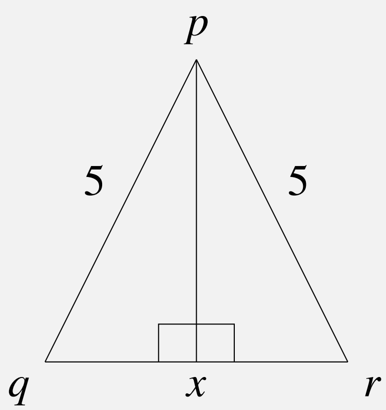


Figure 9.21

Note 9.17 The squares shown at angles $\angle pxq$ and $\angle pxr$ in Figure 9.21 are another piece of non-standard notation that students don't need to memorise for these exams but that are present throughout these notes. They represent angles that are 90°.

A looser relationship between triangles than congruence is when one triangle is a "scaled-up" version of the other.

Definition 9.18 Two triangles are **similar** if all three of their angles are the same.

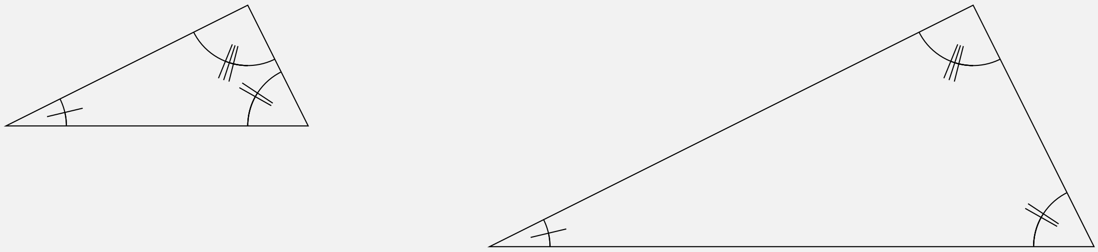


Figure 9.22

Theorem 9.19 Two triangles ABC and DEF are similar if and only if the ratio lengths of their sides are proportional, in order:

$$\frac{|AB|}{|DE|} = \frac{|AC|}{|DF|} = \frac{|BC|}{|EF|}$$

Example 9.20 In the diagram shown below, find x and y.

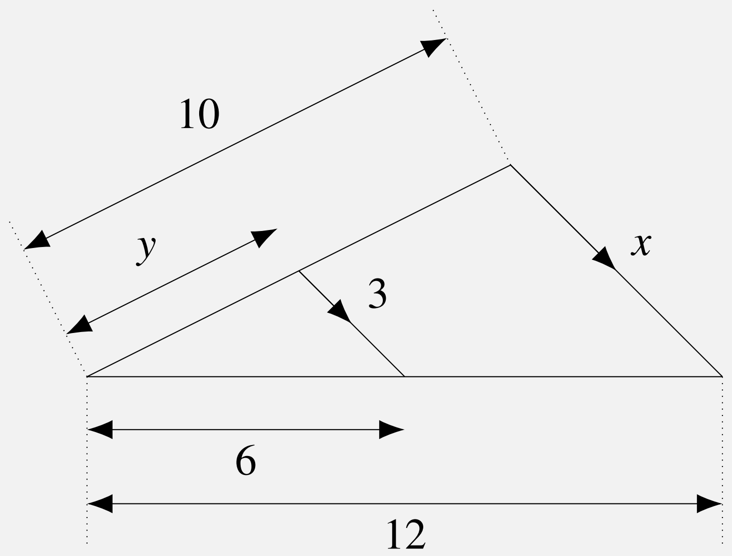


Figure 9.23

Due to corresponding angles and the fact that they share an angle, the triangles are similar so

$$\frac{12}{6} = \frac{x}{3}$$
$$\Rightarrow 6 = x.$$
$$\frac{6}{12} = \frac{y}{10}$$
$$\Rightarrow 5 = y.$$

Question 9.21 In the diagram shown below, show that pq and xy are parallel.

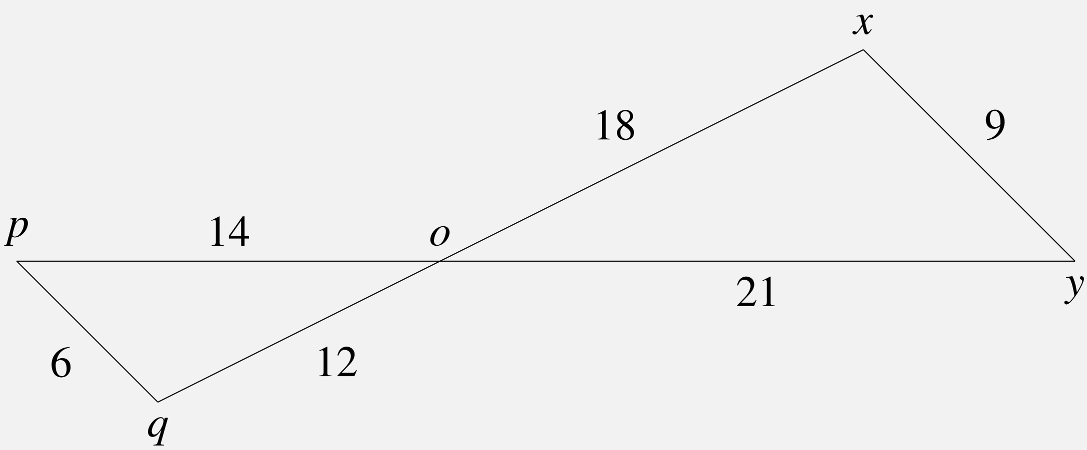


Figure 9.24

9.3 Quadrilaterals & Parallelograms

Definition 9.22

1. A **quadrilateral** is any four sided shape.

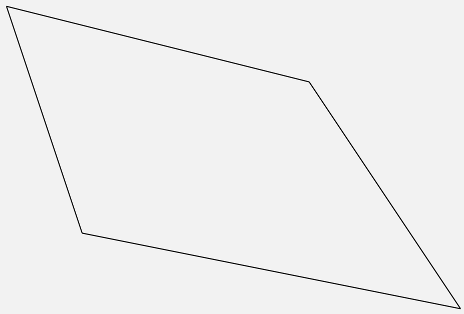


Figure 9.25

2. A cyclic quadrilateral is a quadrilateral where the four vertices can be placed on the circumference of a circle.

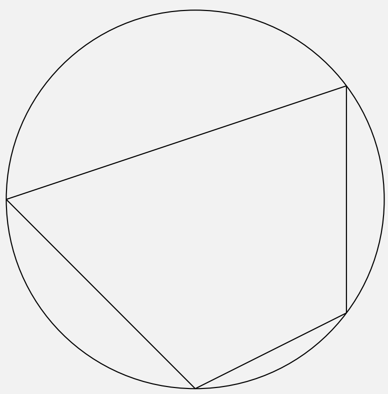
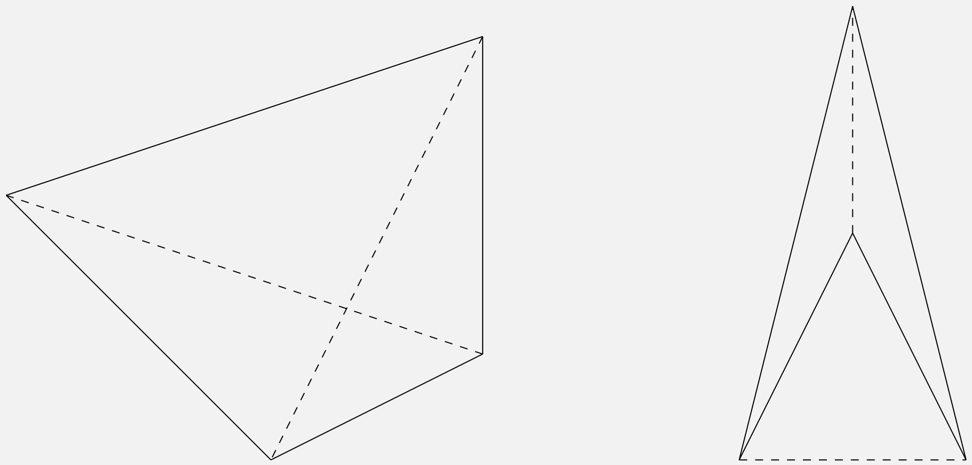


Figure 9.26

3. A convex quadrilateral is one that contains its diagonals.



(a) A convex quadrilateral (b) A non-convex quadrilaterl

Figure 9.27

Theorem 9.23

1. The four angles in any quadrilateral add up to 360°.
2. A quadrilateral is cyclic if and only if its opposite angles add up to 180°.

Definition 9.24 A parallelogram is a shape with four sides and whose opposite sides are parallel.

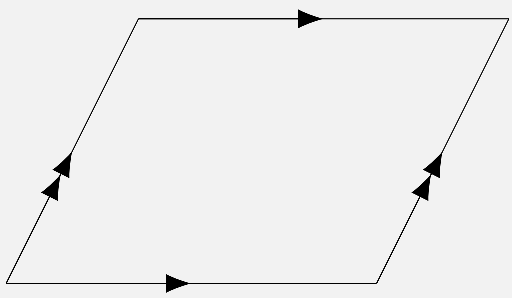


Figure 9.28

Theorem 9.25 Given a convex quadrilateral X, the following statements are equivalent (if one is true they're all true, and if one is false they're all false).

1. X is a parallelogram.
2. The opposite angles of X are equal.
3. The opposite sides of X are equal.
4. The diagonals of X bisect each other.
5. Angles that share a side add up to 180°.

Example 9.26 Calculate the values of the angles A, B, C.

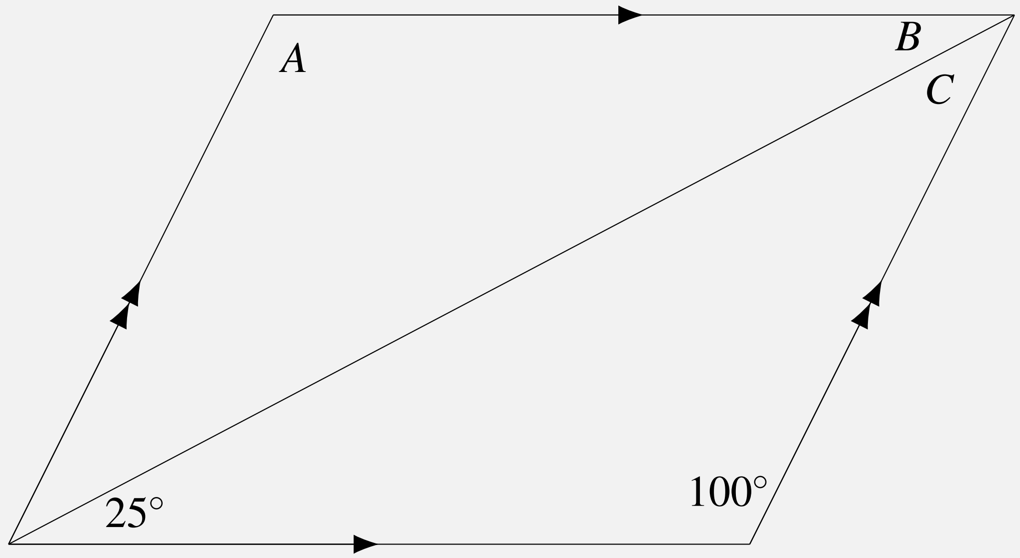


Figure 9.29

$$
\begin{aligned}
A &= 100^\circ \quad \text{(opposite angles)} \\
B &= 25^\circ \quad \text{(alternate angles)} \\
25^\circ + 100^\circ + C &= 180^\circ \quad \text{(triangle)} \\
\Rightarrow C &= 55^\circ.
\end{aligned}
$$

Question 9.27 Calculate the values of the angles A, B, C, D.

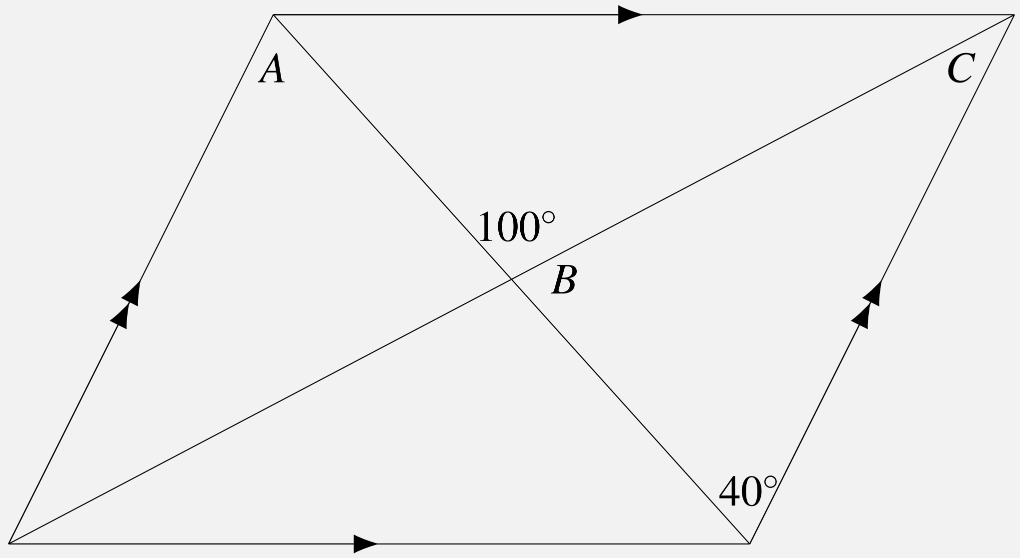


Figure 9.30

Question 9.28 In the parallelogram below all four sides are equal. Find the values of the angles A, B, C. (**Hint:** Use congruence of the triangles).

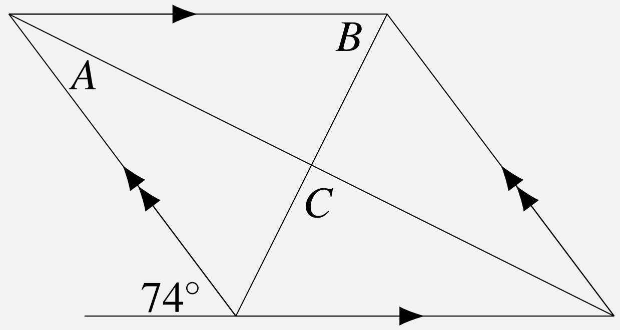


Figure 9.31

9.4 Circles

Theorem 9.29

1. The angle at the centre of a circle facing an arc is twice the angle at any point on the boundary of the circle facing the same arc.

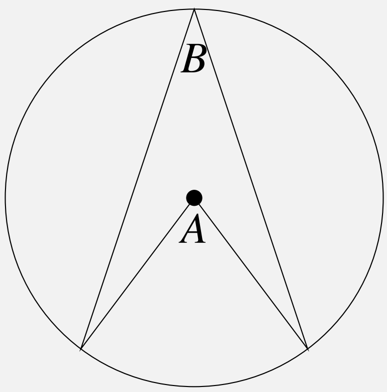


Figure 9.32: $A = 2B$

2. Two angles at different points on the boundary of the circle facing the same arc are equal.

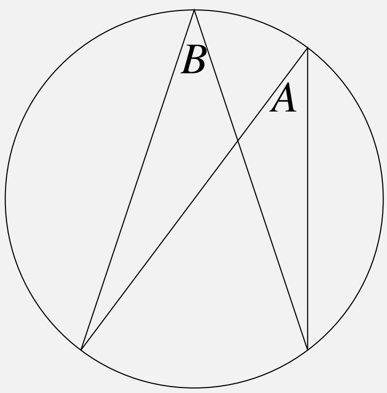


Figure 9.33: $A = B$

3. Triangles on a diameter line are right-angled.

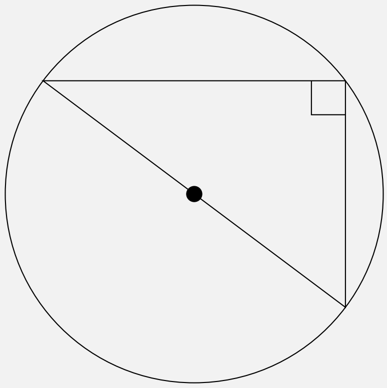


Figure 9.34

4. Radius lines (lines through the centre of the circle) are perpendicular to tangent lines (lines that touch the circle at only one point).

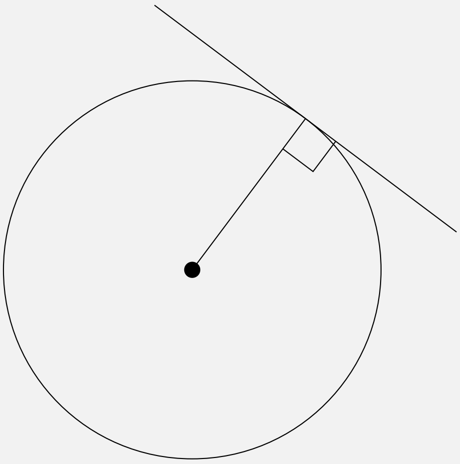


Figure 9.35

5. Radius lines bisect chords (lines that touch the circle at two points) if and only if they intersect them at right angles.

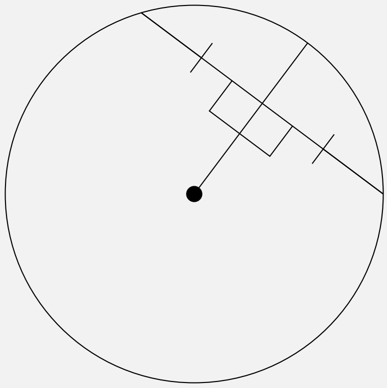


Figure 9.36

Note 9.30 Remember that all radius lines are equal in length, which is useful in proving that certain triangles are congruent or isosceles.

Example 9.31 The black dot is the centre of the circle. Calculate the values of A, B, C.

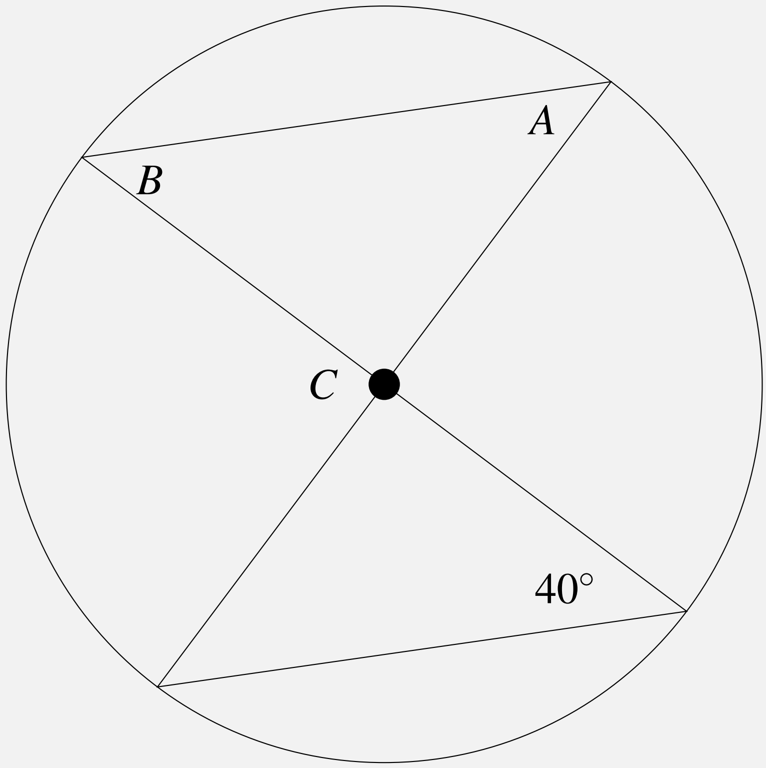


Figure 9.37

$$
\begin{aligned}
A &= 40^\circ \quad \text{(facing same arc)} \\
B &= A \quad \text{(isosceles triangle)} \\
&= 40^\circ \\
C &= 2A \quad \text{(facing same arc)} \\
&= 80^\circ.
\end{aligned}
$$

Question 9.32 The black dot is the centre of the circle. The external lines are tangent lines. Calculate the values of A, B, C.

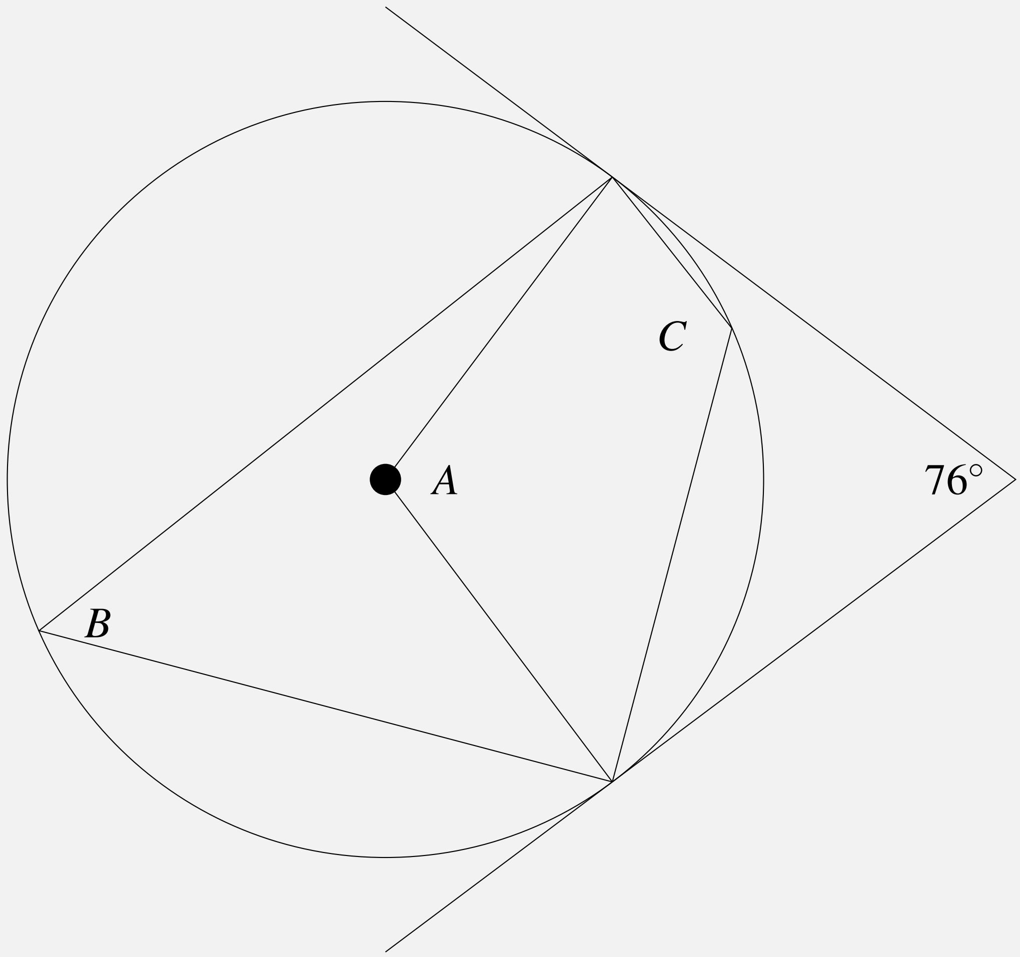


Figure 9.38

Question 9.33 The black dot is the centre of the given circle. Calculate the value of A.

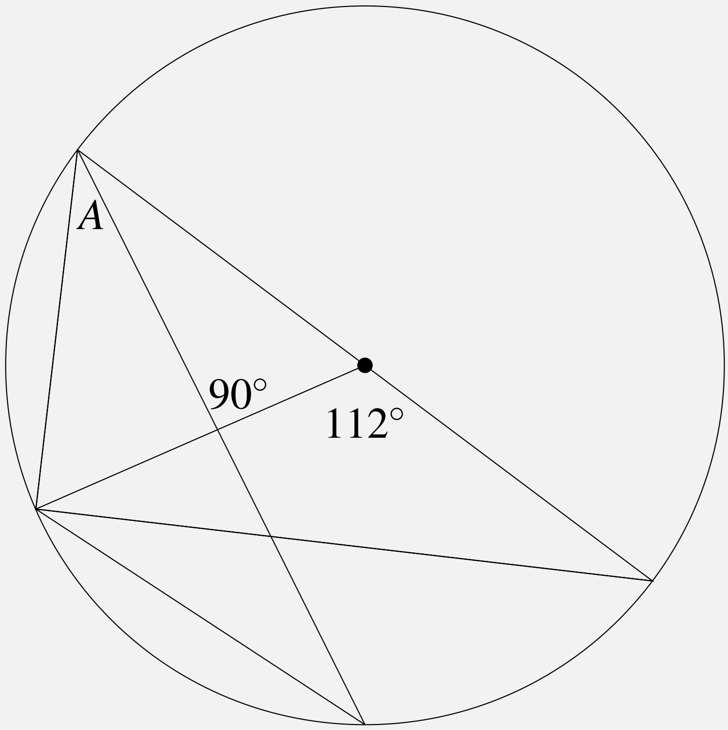


Figure 9.39

Question 9.34 The black dots are the centres of the given circles. Calculate the value of x.

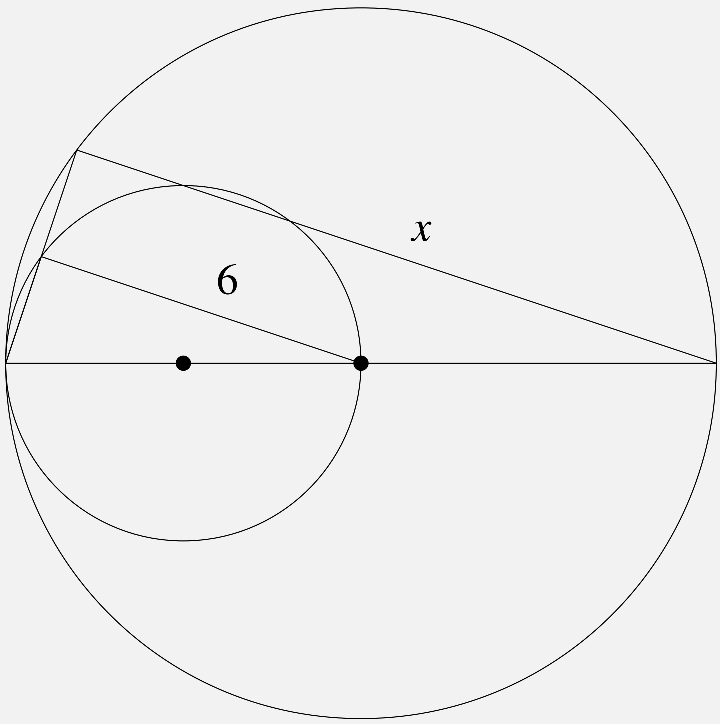


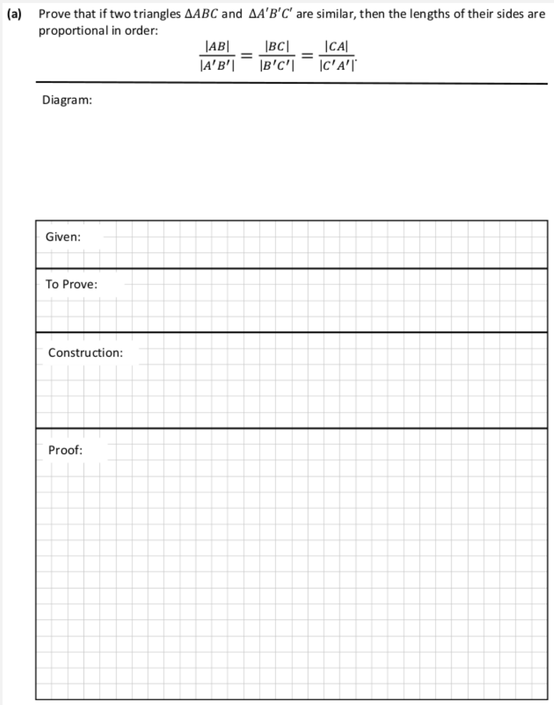
Figure 9.40

9.5 Proofs

At Leaving Cert level there are 3 theorems (Theorems 9.37, 9.39 & 9.42) that you have to be able to prove in the Leaving Cert exam. We have not used these theorems yet to solve any problems, but it is worth knowing that they may be necessary in solving problems we see later in the chapter.

The theorems are written in a particular order; the first theorem can be proved from first principles, while the proof of the second relies on the first and the proof of the third relies on the second.

Note 9.35 When writing a proof of any theorem in this section, there are five parts: Diagram, Given, To Prove, Construction, Proof. In exams these five steps may be explicitly laid out for you.


(a) Prove that if two triangles ΔABC and $\Delta A'B'C'$ are similar, then the lengths of their sides are proportional in order:

$$\frac{|AB|}{|A'B'|} = \frac{|BC|}{|B'C'|} = \frac{|CA|}{|C'A'|}.$$

Diagram:

Given:

To Prove:

Construction:

Proof:

Figure 9.41: An example from the 2021 Leaving Cert exam

- **Diagram** includes a pictorial representation of the situation described in the theorem, for example a triangle if the theorem is about triangles.
- **Given** includes information given in the statement of the theorem, perhaps that two lines are parallel or are of equal length.
- **To Prove** is the statement we are trying to prove, for example $A+B+C = 180°$ where A, B, C would be in the diagram.
- **Construction** is any drawing you do over the original diagram. For example the theorem might be about a triangle but you might want to draw another line to help you in your proof.
- **Proof** is our logical argument. In this section each statement should follow on from some combination of the previous statements. For example, if you show that $A+B = 90°$ and $A+C = 90°$ then it follows that $B = C$.

Note 9.36 In the proofs shown below, another diagram is drawn in the Construction section. This will not be the case in your exam; you will draw over your original diagram (maybe in a different colour) in the Diagram section. The paragraphs written there, however, will be written in the Constructions section of your exam.

Theorem 9.37 If three parallel lines cut off equal line segments on some transversal line then they cut off equal line segments of any other transversal line.

Diagram:

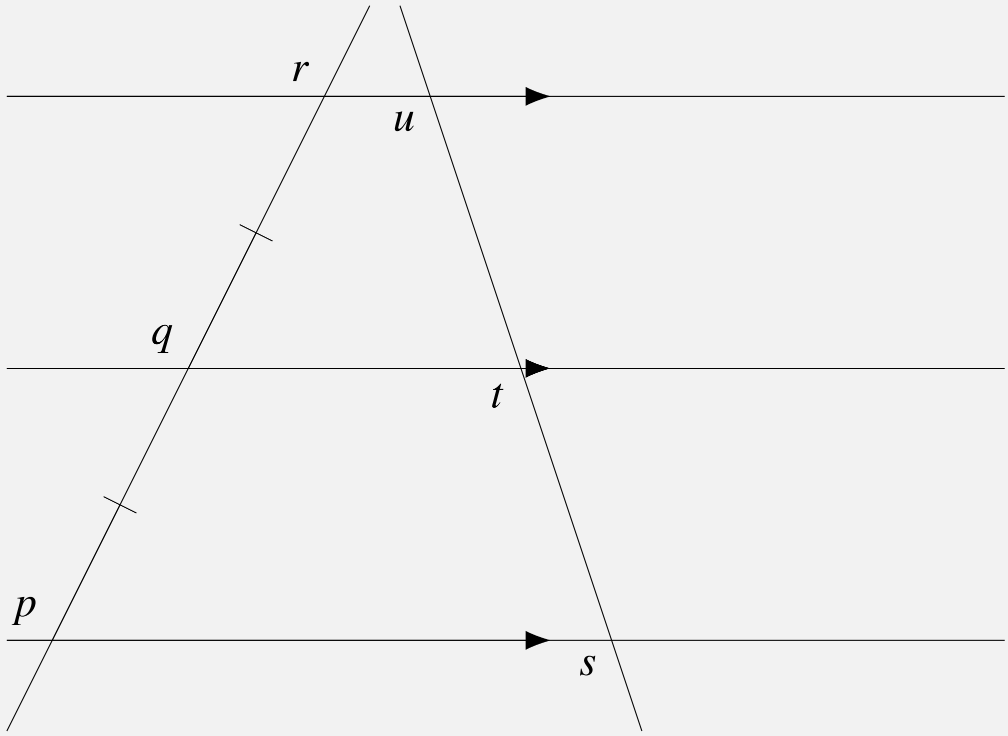


Figure 9.42

To Prove: $|tu| = |st|$.

Given: ps, qt, ru all parallel. $|pq| = |qr|$.

Construction: Draw a line parallel to pr passing through s. Then draw a line parallel to su that passes through the point where the previous line intersects with the top parallel line. Label the extra points v, w, x, y.

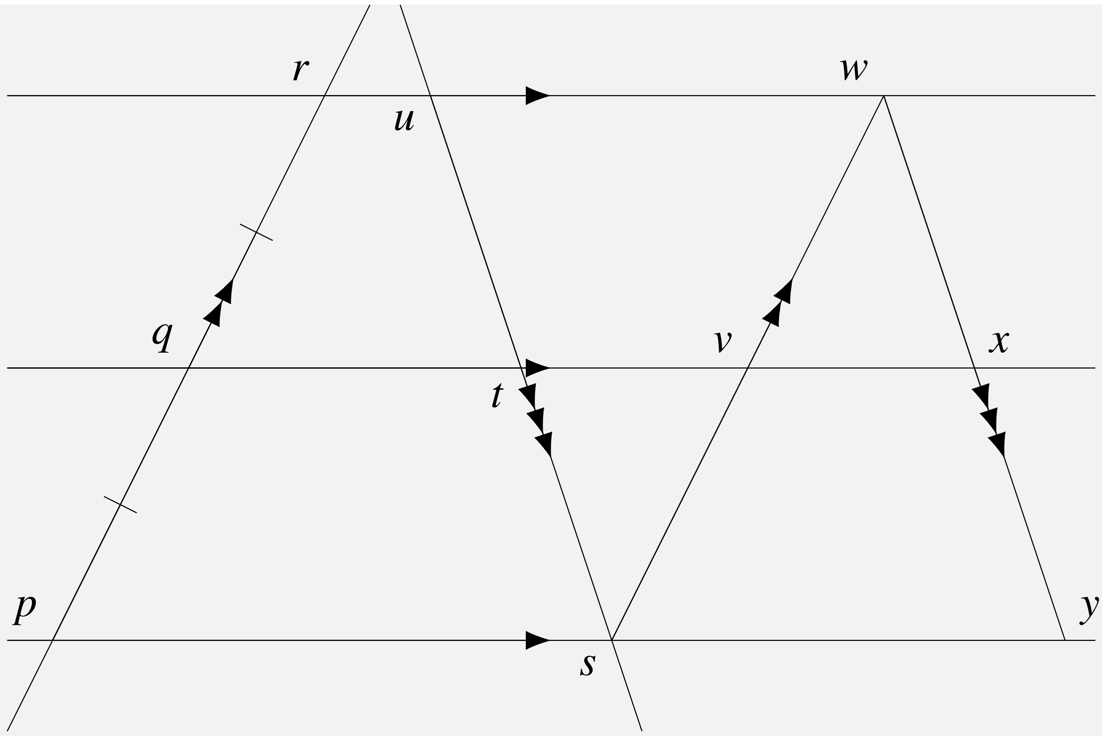


Figure 9.43

Proof:

$$
\begin{aligned}
|vw| &= |qr| \quad \text{(parallelogram)}\\
&= |pq| \quad \text{(given)}\\
&= |sv| \quad \text{(parallelogram)}\\
|\angle tsv| &= |\angle vwx| \quad \text{(alternate angles)}\\
|\angle tvs| &= |\angle wvx| \quad \text{(opposite angles)}\\
\therefore\ \triangle stv \text{ and } \triangle vwx & \text{ are congruent} \quad \text{(ASA)}\\
\Rightarrow |st| &= |wx|\\
&= |tu| \quad \text{(parallelogram).}
\end{aligned}
$$

Note 9.38 The approach of this theorem can be summarised as

1. Draw the line we know is cut equally on the left, sloping upwards, and the line we're trying to prove is cut equally in the middle, sloping downwards.
2. Draw a third line from the second line, parallel to the first.
3. Draw a fourth line from the third line, parallel to the second.
4. Prove the two small triangles are congruent.
5. Finish.

Theorem 9.39 If ABC is a triangle and a line parallel to BC cuts AB in some ratio $s:t$, then it cuts AC in the same ratio.

Diagram: In this diagram all letters represent points.

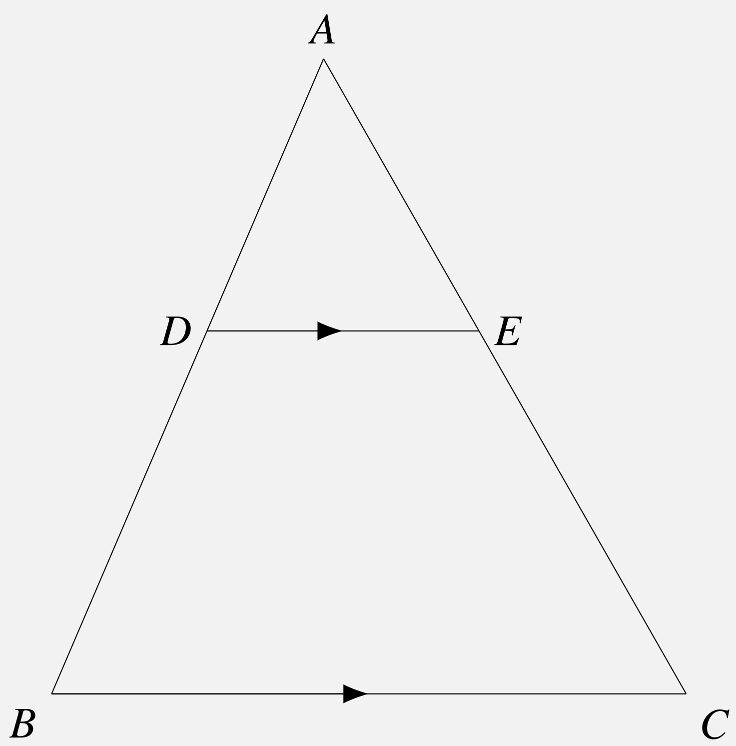


Figure 9.44

To Prove: $|AE|:|EC| = s:t$.

Given: BC, DE parallel. $|AD|:|DB| = s:t$.

Construction: Divide AD into s equal parts and divide BD into t equal parts. Draw lines from these points parallel to DE.

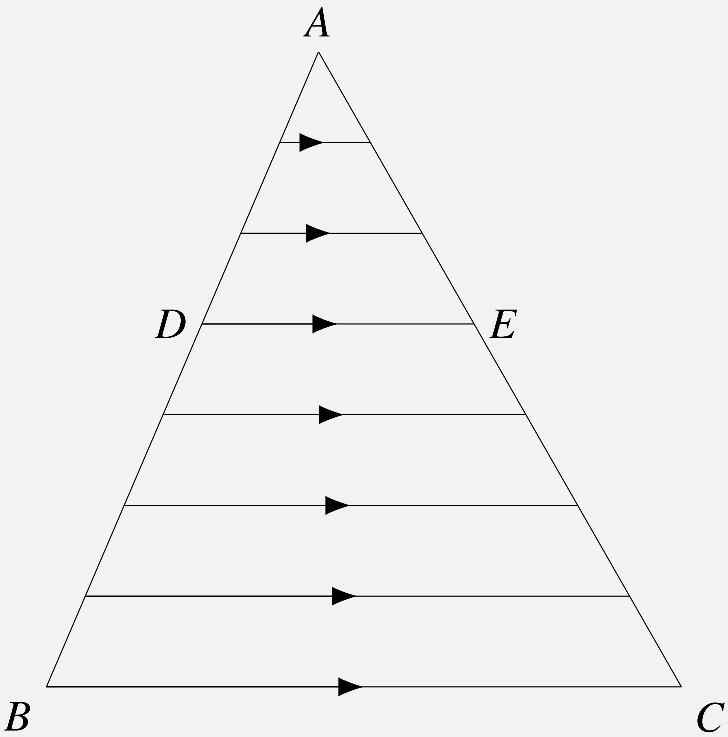


Figure 9.45

Proof: As each line segment of AB is equal, each line segment of AC is equal (by Theorem 9.37). Letting the length of these line segments be k, $|AE| = ks$ and $|EC| = kt$. Therefore

$$\begin{aligned}\frac{|AE|}{|EC|} &= \frac{ks}{kt}\\ &= \frac{s}{t}\\ \Rightarrow |AE| : |EC| &= s : t.\end{aligned}$$

Note 9.40 The approach of this theorem can be summarised as

1. Draw the triangle with the parallel line DE with the left line of the triangle split in the ratio $s : t$.
2. Split the left line into $s+t$ equal pieces and draw parallel lines coming out of them.
3. Prove the right line of the triangle is also split into equal pieces of length k.
4. Prove $\frac{\text{Right Line Top}}{\text{right Line Bottom}} = \frac{s}{t}$.

Note 9.41 When reproducing these theorems in an exam there is no need to reference the theorems as "Theorem 9.37" or whatever (those numbers are specific to my notes anyway).

Theorem 9.42 If two triangles are similar then the lengths of their sides are proportional, in order:

$$\frac{|AB|}{|DE|} = \frac{|AC|}{|DF|} = \frac{|BC|}{|EF|}$$

Diagram: In this diagram all letters represent points.

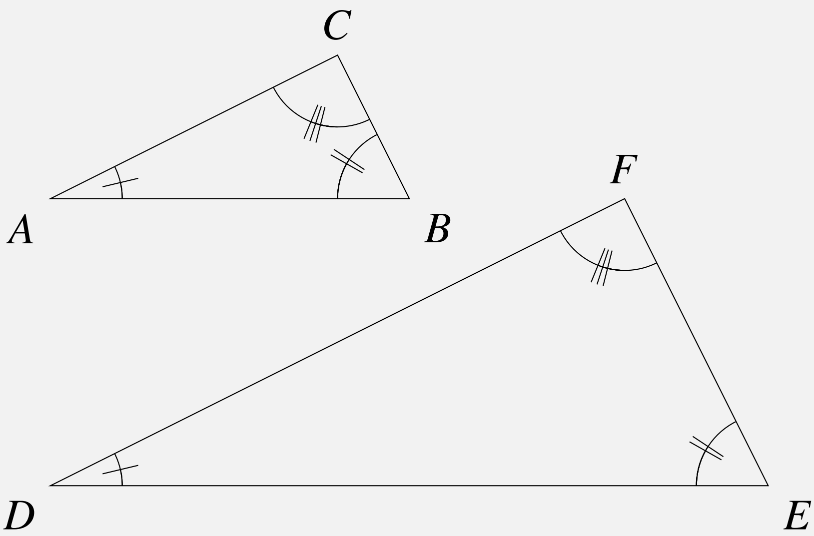


Figure 9.46

To Prove: $\frac{|AB|}{|DE|} = \frac{|AC|}{|DF|} = \frac{|BC|}{|EF|}$

Given: angles shown are equal.

Construction: Mark a point X on DE so that $|DE| = |AB|$. Mark a point Y on DF so that $|DY| = |AC|$.

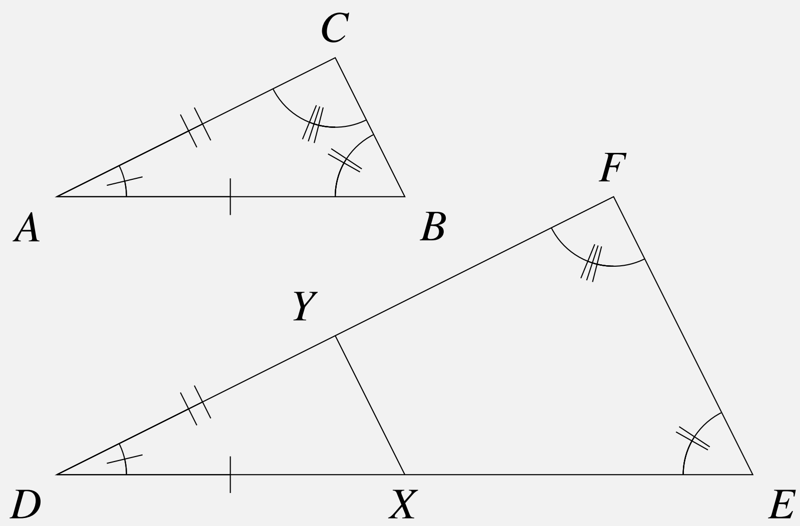


Figure 9.47

Proof:

$$\triangle ABC \text{ and } \triangle DXY \text{ are congruent} \quad \text{(SAS)}$$
$$\Rightarrow |\angle DXY| = |\angle ABC|$$
$$\Rightarrow XY \text{ is parallel to } EF.$$

Therefore XY cuts DF in the same ratio that it cuts DE in (by Theorem 9.39). Call this ratio $s:t$. Therefore

$$\frac{|DX|}{|DE|} = \frac{s}{s+t}$$
$$= \frac{|DY|}{|DF|}$$
$$\Rightarrow \frac{|AB|}{|DE|} = \frac{|AC|}{|DF|}.$$

Similarly

$$\frac{|AB|}{|DE|} = \frac{|BC|}{|EF|}.$$

Note 9.43 The approach of this theorem can be summarised as

1. Draw the two similar triangles.
2. Pick an angle of the larger triangle. Draw a triangle inside the larger one with the same length lines as the lines in the smaller triangle making this angle.
3. Prove this new triangle is congruent to the smaller one.
4. Argue the line cutting the larger triangle is parallel to the other line of the triangle.
5. Apply the previous theorem to set up the ratios.
6. State its the same for other pairs of sides.

9.6 General Geometry Problems

Although it was necessary to learn about parallelograms, circles etc. in isolation, most realistic geometry problems combine these shapes and the theorems from Section 9.5. The following are a collection of problems (some of which are from old syllabus past Leaving Cert exams). They are purposely not separated by section as we studied them in these notes. Unlike most problems marked "Question" in these notes, solutions are given at the end of this section.

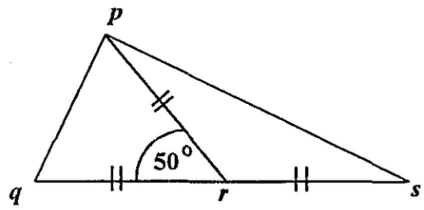
Question 9.44

$|pr| = |qr| = |rs|$ and $|\angle prq| = 50°$.

Find

(i) $|\angle pqr|$

(ii) $|\angle psr|$.

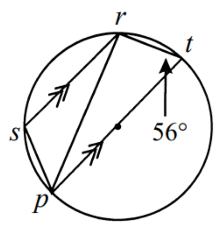


Question 9.45

In the diagram, $[pt]$ is a diameter of the circle. sr is parallel to pt and $|\angle ptr| = 56°$.

(i) Write down the value of $|\angle prt|$.

(ii) Find the value of $|\angle prs|$.

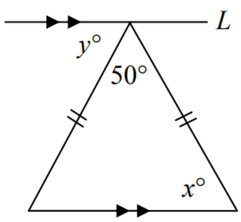


Question 9.46

In the diagram, the line L is parallel to the base of the isosceles triangle.

(i) Find x.

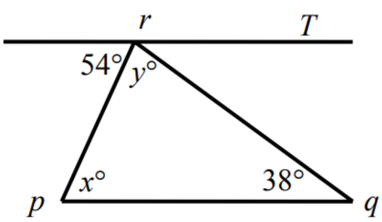
(ii) Find y.



Question 9.47

The line T passes through r and is parallel to pq.

Calculate the value of x and the value of y in the diagram.

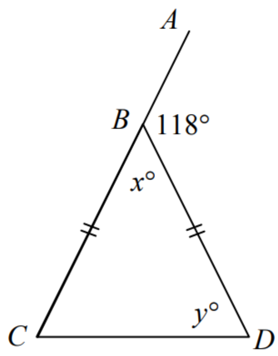


Question 9.48

In the diagram,
$|BC| = |BD|$ and $|\angle ABD| = 118°$.

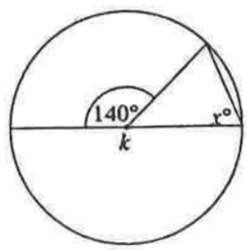
(i) Find x.

(ii) Find y.



Question 9.49

k is the centre of the circle.
Find the value of x.



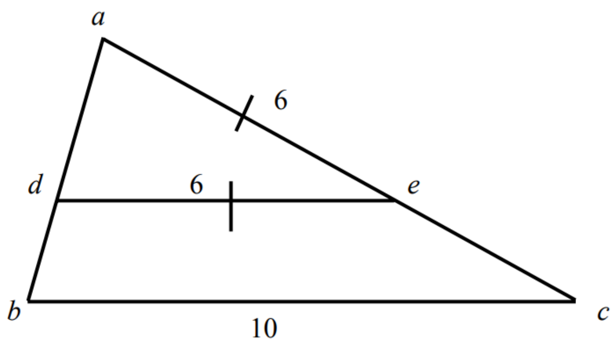
Question 9.50

In the triangle abc, $bc \parallel de$, $|ae| = |de| = 6$ and $|bd| = ½\ |ce|$.
$|bc| = 10$.

(i) ✍ Find $|ce|$.

(ii) ✍ Find $|ad|$.

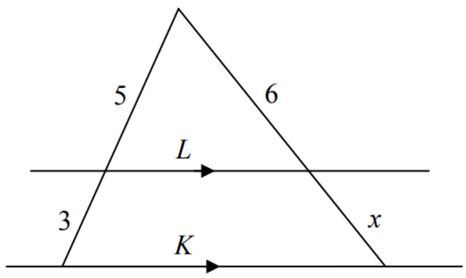
(iii) ✍ Find $|ab|$.



Question 9.51

In the diagram $L \parallel K$.

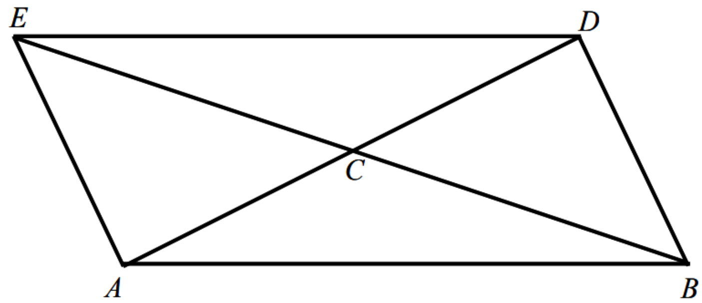
Find the value of x.



Question 9.52

The quadrilateral $ABDE$ has diagonals $[AD]$ and $[BE]$ intersecting at C.

C is the midpoint of both $[AD]$ and $[BE]$.



(i) ✍ Prove that ΔECD is congruent to ΔACB.

(ii) ✍ Hence, prove that $ABDE$ is a parallelogram.

Question 9.53

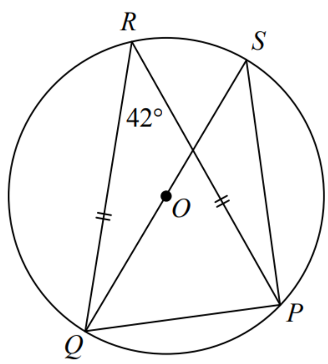
In the diagram, O is the centre of the circle and $|\angle PRQ| = 42°$.
$[QS]$ is a diameter and $|RQ| = |PR|$.

(i) Find $|\angle PSQ|$.

(ii) Find $|\angle SQP|$.

(iii) Find $|\angle QPR|$.

(iv) Find $|\angle RQS|$.



Question 9.54

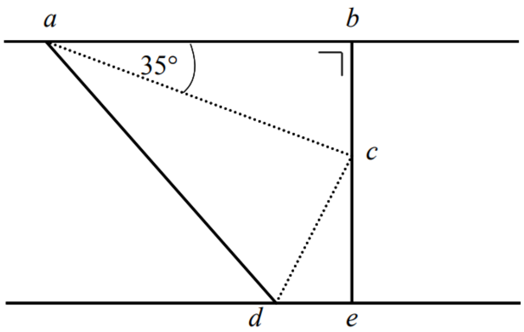
ab is parallel to *de*,

ac bisects $\angle bad$,

dc bisects $\angle ade$,

be is perpendicular to *ab* and

$|\angle bac| = 35°$.



(i) ✍ Find $|\angle ade|$.

(ii) ✍ Find $|\angle acd|$.

(iii) ✍ Prove that the triangles *adc*, *abc* and *cde* are equiangular.

(iv) ✍ Given that $|ab| = 5$ and $|bc| = 3{\cdot}5$, write $|de| : |ec|$ in the form $m : n$, where $m, n \in \mathbf{N}$.

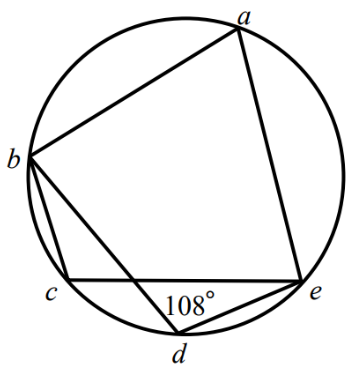
Question 9.55

a, *b*, *c*, *d* and *e* are points on a circle and $|\angle bde| = 108°$.

Find **(i)** ✍ $|\angle bae|$,

(ii) ✍ $|\angle bce|$,

giving a reason for your answer in each case.



Question 9.56

pt is a tangent to a circle of centre *c*.

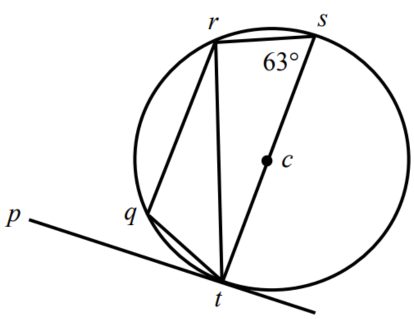
[*ts*] is a diameter of the circle.

r is a point on the circle such that $|\angle tsr| = 63°$.

(i) Find $|\angle ptr|$.

q is a point on the circle such that $qr \parallel ts$.

(ii) Find $|\angle trq|$.

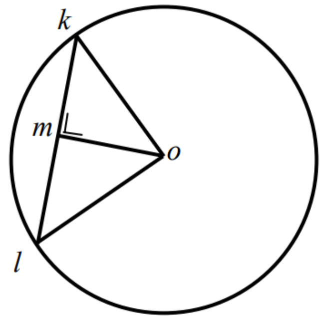


Question 9.57

A circle, centre o, has a radius of length 17.

$[lk]$ is a chord of length 30.

m is a point on $[lk]$ and lk is perpendicular to mo.

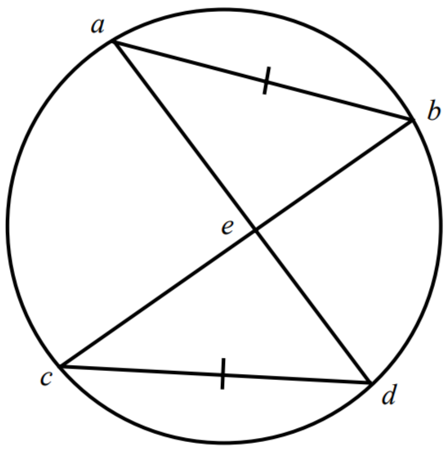


✍ Write down the length of $[km]$, giving a reason for your answer.

Question 9.58

$[ab]$ and $[cd]$ are chords of the circle as shown and $|ab| = |cd|$.

The chords $[ad]$ and $[bc]$ intersect at the point e.



(i) State why $|\angle bad| = |\angle bcd|$.

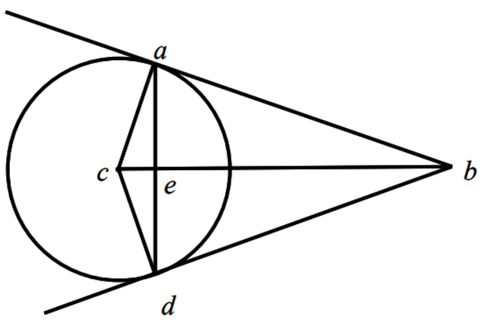
(ii) ✍ Prove that the triangles bae and dce are congruent.

(iii) ✍ Prove $|ad| = |bc|$.

Question 9.59

ba and bd are tangents to the circle of centre c.

$[bc]$ intersects the chord $[ad]$ at the point e.



(i) ✍ Prove that $\Delta\, abc$ is congruent to $\Delta\, dbc$.

(ii) ✍ Hence, prove that $[bc]$ bisects the chord $[ad]$.

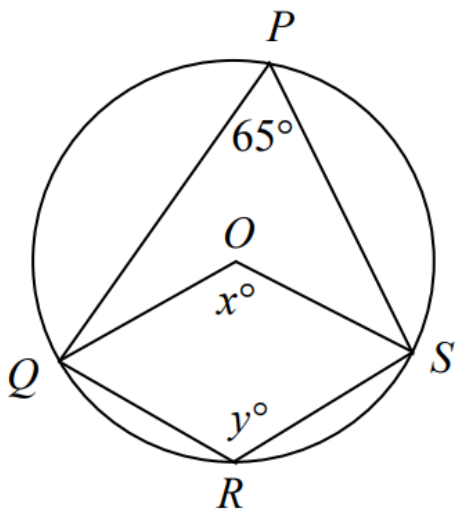
Question 9.60

The points P, Q, R and S lie on a circle, centre O.

$|\angle SPQ| = 65°$.

(i) Find the value of x.

(ii) Find the value of y.



Solutions:

44. (i) $65°$
    (ii) $25°$
45. (i) $90°$
    (ii) $34°$
46. (i) $65°$
    (ii) $65°$
47. $x = 54°$, $y = 88°$
48. (i) $62°$
    (ii) $59°$
49. $70°$
50. (i) 4
    (ii) 3
    (iii) 5
51. 3.6
52. (i) As C is the midpoint of both lines, $|EC| = |CB|$ and $|AC| = |CD|$. $|\angle ECD| = |\angle ACB|$ (opposite angles) so they are congruent by SAS.
    (ii) $|\angle CAB| = |\angle EDC|$ by congruence so ED, AB are parallel. Similarly $|\angle CEA| = |\angle CBD|$ so AE, BD are parallel.
53. (i) $42°$
    (ii) $48°$
    (iii) $69°$
    (iv) $21°$
54. (i) $110°$
    (ii) $55°$
    (iii) Triangles adc, abc both have a $35°$ and $55°$ angle from earlier work. Therefore they both have a $90°$ angle and are equiangular. Similarly, from previous work cde has a $35°$ angle. It has a $90°$ angle (given) and so is equiangular to the other two.
    (iv) 7 : 10
55. (i) $72°$
    (ii) $108°$
56. (i) $63°$
    (ii) $27°$
57. 15 as radius lines bisect chords.
58. (i) Standing on same arc.
    (ii) $|\angle abc| = |\angle adc|$ for the same reason, so congruent by ASA.
    (iii) By congruence $|ce| = |ae|$ and $|be| = |de|$ so $|ad| = |bc|$.
59. (i) $|ac| = |cd|$ (radius), $|\angle cab| = |\angle cdb| = 90°$ (tangent meets radius) and $|bc| = |bc|$ so congruent by RSH.
    (ii) By congruence $|ab| = |bd|$ and $|\angle abe| = |\angle dbe|$, and $|be| = |be|$ so aeb, deb are congruent as well making $|ae| = |ed|$.
60. (i) $130°$
    (ii) $115°$

9.7 Summary

Although it serves as a foundation for Constructions (Chapter 10) Trigonometry (Chapter 11) and Coordinate Geometry (Chapters 13 and 14), Geometry shouldn't be dismissed as a foundational topic. There have been many Leaving Cert exams in the past purely based on geometry. They can be spotted by the lack of numerical measurements in the diagram. Many of the questions given in Section 9.6, and the Homework and Revision sections are from old syllabus (pre-2014) Leaving Cert exams.

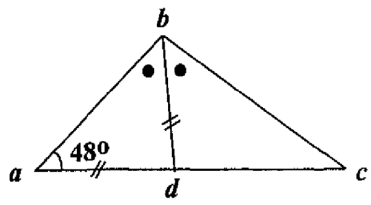
9.8 Homework

General Geometry Problems

1.

In the triangle abc, $|ad| = |bd|$,
$|\angle abd| = |\angle dbc|$ and $|\angle dab| = 48°$.

Find $|\angle dcb|$.

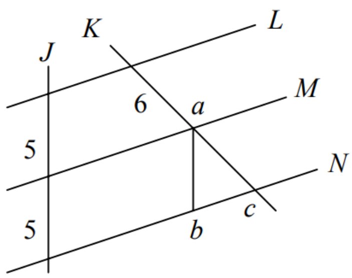


2.

In the diagram, L, M and N are parallel lines. They make intercepts of the indicated lengths on J and K. ab is parallel to J.

(i) Write down the length of $[ab]$.

(ii) Write down the length of $[ac]$.

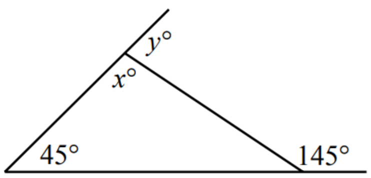


3.

In the diagram, two sides of the triangle are produced.

(i) Find x.

(ii) Find y.

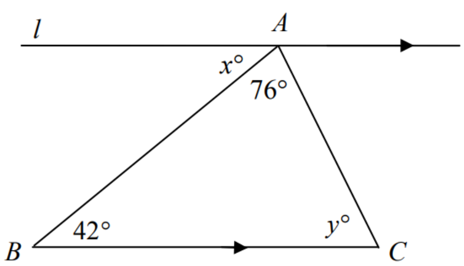


4.

In the diagram, the line l passes through the point A and is parallel to BC.

(i) Find x.

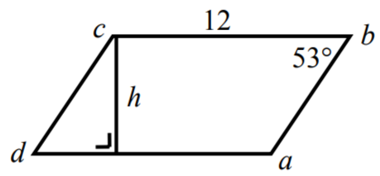
(ii) Find y.



5.

In the parallelogram *abcd*,
$|\angle abc| = 53°$ and $|bc| = 12$ cm.

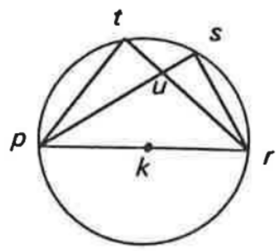
Find $|\angle bcd|$.



6.

k is the centre of the circle.
$|\angle tps| = 30°$.

Calculate $|\angle sur|$.



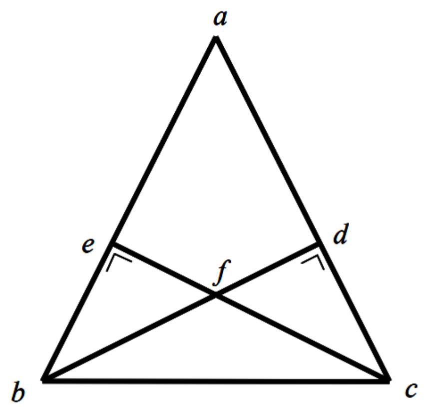
7.

The triangle *abc* is an isosceles triangle, with $|ab| = |ac|$ and $|\angle bec| = |\angle cdb| = 90°$.

The lines *ec* and *bd* intersect at *f*.

(i) ✍ Prove $|\angle dbc| = |\angle ecb|$.

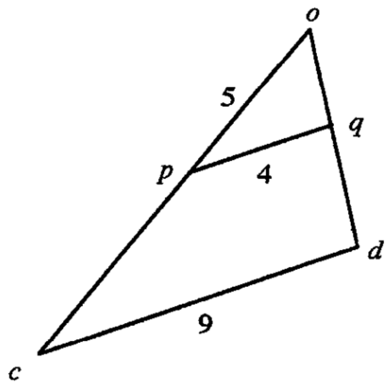
(ii) ✍ Prove $|ef| = |fd|$.



8.

The triangle *ocd* is the image of the triangle *opq* under the enlargement, centre *o*, with $|pq| = 4$, $|op| = 5$ and $|cd| = 9$.

Find $|pc|$.



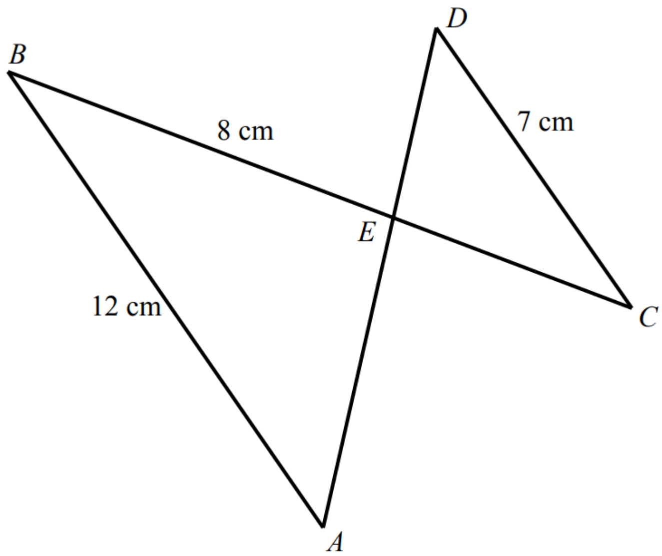
9.

AB is parallel to *CD*. *BC* and *AD* intersect at the point *E*.

(i) ✍ Prove that the triangles *ABE* and *CDE* are equiangular.

$|AB| = 12$ cm, $|BE| = 8$ cm and $|CD| = 7$ cm.

(ii) ✍ Find $|EC|$ correct to one decimal place.

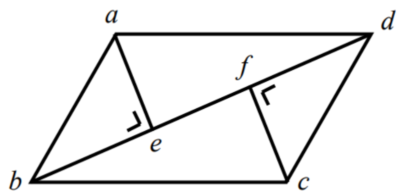


10.

abcd is a parallelogram.

ae and *cf* are perpendicular to *bd* as shown.

Prove the triangles *abe* and *dcf* are congruent.



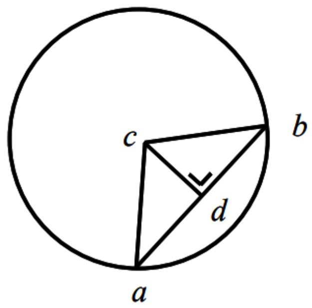
11. **Hint: Needs Pythagoras' Theorem**

A circle, centre *c*, has a chord [*ab*] of length 8.

d is a point on [*ab*] and *cd* is perpendicular to *ab*.

$|cd| = 3$.

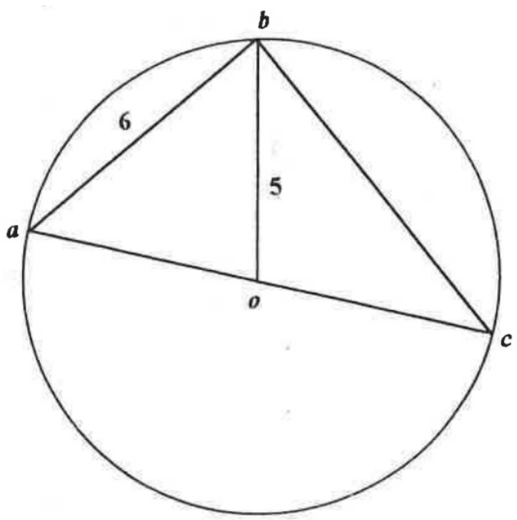
✍ Find the length of a diameter of the circle.



12. **Hint: Needs Pythagoras' Theorem**

The centre of the circle is *o*,
| *ab* | = 6 and | *bo* | = 5.

Find | *bc* |.



13.

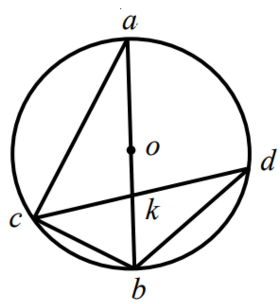
[*ab*] is a diameter of the circle of centre *o*.

c and *d* are points on the circle.

[*ab*] and [*cd*] intersect at *k*.

$|\angle cdb| = 38°$ and $|\angle ckb| = 80°$.

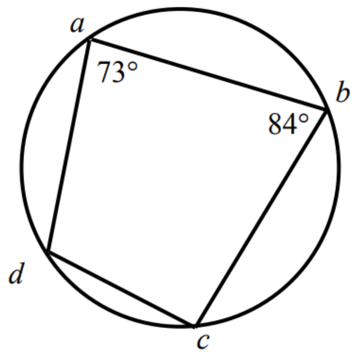
Write down $|\angle cab|$ and then find $|\angle dcb|$.



14.

abcd is a cyclic quadrilateral.

✍ Given that $|\angle dab| = 73°$ and $|\angle abc| = 84°$, find $|\angle adc|$ and $|\angle bcd|$.

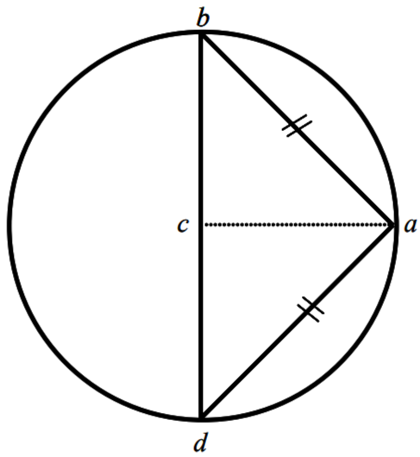


15.

[*bd*] is the diameter of the circle, *c* is the centre of the circle and $|ba| = |ad|$.

Find **(i)** ✍ $|\angle adb|$,

(ii) ✍ $|\angle dac|$.

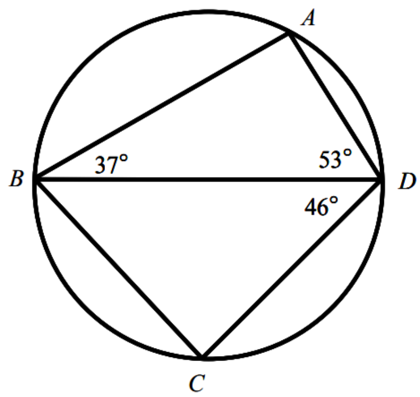


16.

A, *B*, *C* and *D* are points on the circle as shown. $|\angle ABD| = 37°$ and $|\angle ADB| = 53°$.

(i) ✍ Explain why [*BD*] is a diameter of the circle.

(ii) ✍ Given that $|\angle BDC| = 46°$, find $|\angle CBD|$.



17. Prove Theorems 9.37, 9.39 and 9.42.

9.9 Homework Solutions

1. 36°
2. (i) 5
   (ii) 6
3. (i) 100°
   (ii) 80°
4. (i) 42°
   (ii) 62°
5. 127°
6. 60°
7. (i) $|\angle ebc| = |\angle dcb|$ (isosceles) so $|\angle dbc| = |\angle ecb|$.
   (ii) As $|\angle ebc| = |\angle bdc|$, $|\angle bdc| = |\angle ebc|$ and $|bc| = |bc|$ triangles ebc and bdc are congruent. Therefore $|ec| = |bd|$. As $|cf| = |bf|$ (isosceles triangle bfc) $|ef| = |fd|$.
8. 6.25
9. (i) $|\angle ABE| = |\angle EDC|$, $|\angle BAE| = |\angle ECD|$ (alternate angles) and $|\angle BEA| = |\angle DEC|$ (opposite angles).
   (ii) 4.7
10. $|ab| = |cd|$ (parallelogram), $|\angle abe| = |\angle fdc|$ (alternate) and the right angle mean abe, cdf are congruent by RSH.
11. 5
12. 8
13. (i) 38°
    (ii) 48°
14. (i) 96°
    (ii) 107°
15. (i) 45°
    (ii) 45°
16. (i) Because $|\angle BAD| = 90°$, so if O is the centre of the circle $|\angle BOD| = 180°$.
    (ii) 44°

9.10 Harder Problems

The problems in this section are designed to be harder than homework problems, but are not designed to be Leaving Cert problems. They're arguably not even helpful in revising for the Leaving Cert exam, and are more for advanced students to test their general understanding of the work in this section at a high level.

It is difficult to think about geometry at a high level without taking a deep dive into what our assumptions, which is quite tedious and not very illuminating to a Leaving Cert student. Instead, look ahead to Chapter 10.

Can you prove why Constructions 10.16-10.19 work? Why do they give the construction you are looking for? You can use any theorems from this geometry chapter.

9.11 Harder Problems Solutions

16. If we bisest AB at the midpoint M with the line l, consider any point P on l.

$$\begin{aligned}|\angle PMA| &= 90^\circ \\ &= |\angle PMB|, \\ |AM| &= |BM|, \\ \text{and } |MP| &= |MP|\end{aligned}$$

so triangles AMP, BMP are congruent by RSH. Therefore $|AP| = |BP|$; any point on P is equidistant from A and B. The same would be true of the line that bisects BC. Therefore a point on the intersection of the line that bisect AB and the line that bisects BC would be equidistant from all three. Therefore a circle drawn with centre at this point containing A (on its boundary) would contain all three.

17. Consider any point P on the line that bisects $\angle CAB$. Draw a line from P to AC, perpendicular to AC, and let X be the point of intersection. Similarly, draw a line from P to AB, perpendicular to AB, and let Y be the point of intersection.

$$\begin{aligned}|\angle CAP| &= |\angle BAP|, \\ |\angle AXP| &= 90^\circ \\ &= |\angle AYP|, \\ \text{and } |AP| &= |AP|\end{aligned}$$

so triangles AXP, AYP are congruent by RSH. Therefore $|PX| = |PY|$ and so any circle with centre P that is tangent to AC will also be tangent to AB. This is true of any point on the line that bisects $\angle CAB$. The same is true of the line that bisects $\angle BCA$. Therefore if a circle is drawn at the point of intersection of two of these lines that is tangent to one line it will be tangent to all three.

18. Let the line drawn be AB and the point of intersetion of the arcs be X. Then $|AB| = |AX| = |BX|$ and so triangle ABX is equilateral.

19. Let the intersection of the first arcs with the circle be XY, and let the point of intersection of the second two arcs be O. Draw a line XY and a line PO. Let the point of intersection of these lines be M.

First,

$$\begin{aligned}|OX| &= |OY| \text{ (radius lines)}, \\ |PX| &= |PY|, \\ |OP| &= |OP|\end{aligned}$$

and so triangles OPX, OPY are congruent by SSS. Therefore

$$\begin{aligned}|\angle XOP| &= |\angle|YOP|, \\ |OX| &= |OY| \text{ (radius lines)}, \\ |OM| &= |OM|\end{aligned}$$

and so triangles OMX, OMY are congruent by SAS. Therefore M bisects XY. Any line drawn from the centre of the circle is a radius line and so would perpendicularly bisect XY, and so would overlap with OP. Therefore the centre of the circle lies somewhere on the extended version of the line segment OP.

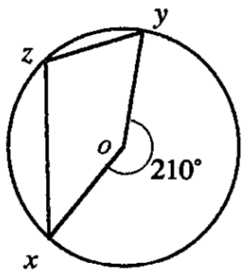
9.12 Revision

General Geometry Problems

1.

In the diagram, o is the centre of the circle and $|\angle xoy| = 210°$.

Find $|\angle xzy|$.



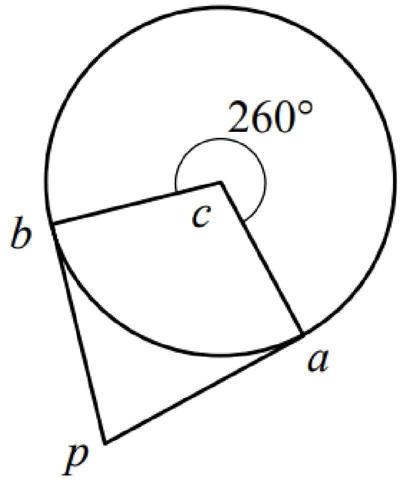
2.

In the diagram, the lines pa and pb are tangents to the circle at a and b respectively. c is the centre of the circle.

Find

(i) $|\angle bca|$

(ii) $|\angle apb|$.

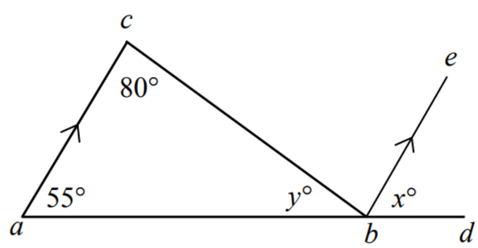


3.

In the diagram, ac is parallel to be, $|\angle bca| = 80°$ and $|\angle cab| = 55°$.

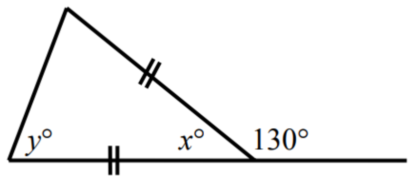
(i) Find x.

(ii) Find y.



4.

Calculate the value of *x* and the value of *y* in the diagram.

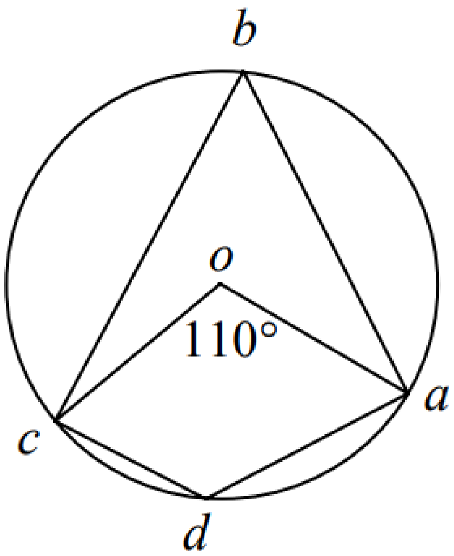


5.

The points *a*, *b*, *c* and *d* lie on a circle, centre *o*.
$|\angle aoc| = 110°$.

(i) Find $|\angle abc|$.

(ii) Find $|\angle cda|$.

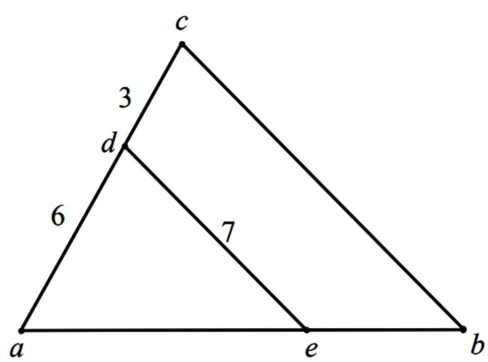


6.

In the triangle *abc*, *de* is parallel to *cb*.

$|ad| = 6$, $|dc| = 3$ and $|de| = 7$.

✍ Find $|cb|$.

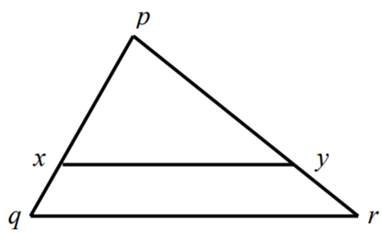


7.

In the triangle pqr, xy is parallel to qr.

$|pq| = 14$ cm, $|qr| = 21$ cm and $|xq| = 4$ cm.

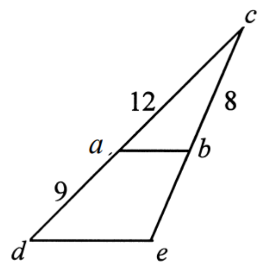
Find $|xy|$.



8.

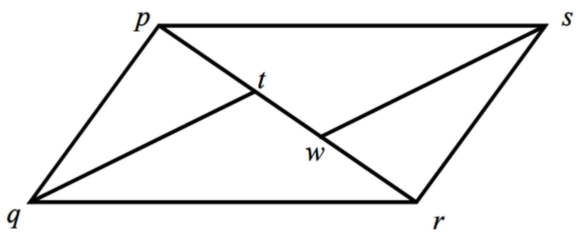
The triangle cde is the image of the triangle cab under an enlargement with centre c.
$|ca| = 12$, $|ad| = 9$ and $|cb| = 8$.

Find $|be|$.



9.

In the parallelogram $pqrs$, the points t and w are on the diagonal $[pr]$ such that $|\angle pqt| = |\angle wsr|$.

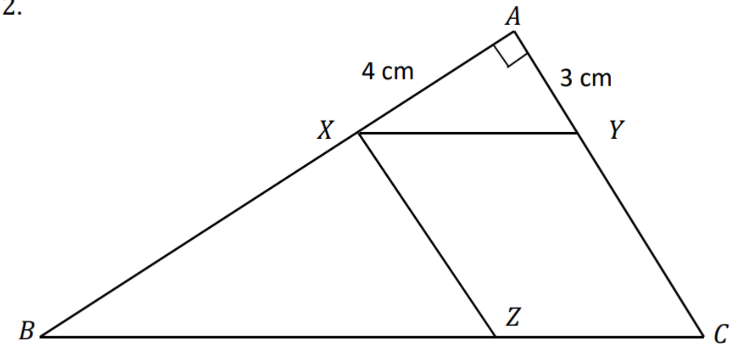


(i) ✍ Prove that $|pt| = |wr|$.

(ii) ✍ Hence, or otherwise, show that the triangles psw and qtr are congruent.

10. **Hint: Requires Pythagoras' Theorem**

In the triangle ABC shown below:
$|\angle CAB| = 90°$, $|AX| = 4$ cm, $|AY| = 3$ cm, $XY \parallel BC$, $XZ \parallel AC$,
and $|AX| : |XB| = 1 : 2$.
Find $|BZ|$.



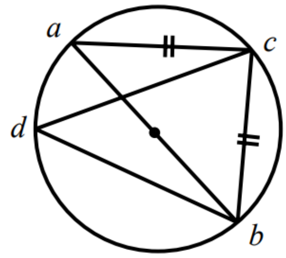
11. **Hint: Requires Pythagoras' Theorem**

a, d, b, c are points on a circle, as shown.
$[ab]$ is a diameter of the circle.
$|ab| = 12$ cm and $|ac| = |cb|$.

(i) ✍ Write down $|\angle bca|$, giving a reason for your answer.

(ii) ✍ Find $|\angle cdb|$.

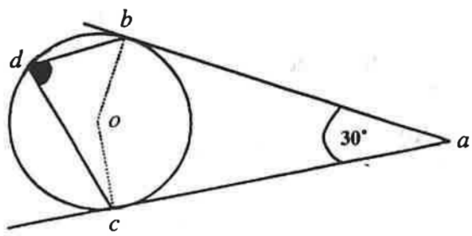
(iii) ✍ Find $|bc|$.



12.

ab and ac are tangents to the circle at b and c, respectively. The centre of the circle is o and d is a point on the circle.

If $|\angle bac| = 30°$, find $|\angle bdc|$.



13.

a, d, b, c are points on a circle, as shown.

o is the centre of the circle.

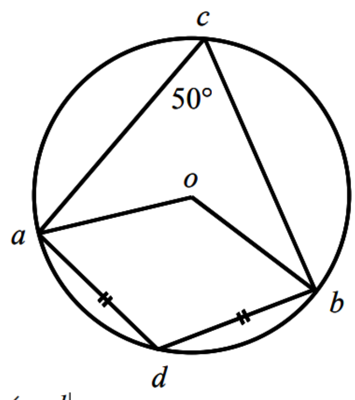
$|\angle acb| = 50°$ and $|ad| = |db|$.

Find

(i) $|\angle aob|$

(ii) ✍ $|\angle adb|$

(iii) ✍ By joining a to b, or otherwise, find $|\angle oad|$.



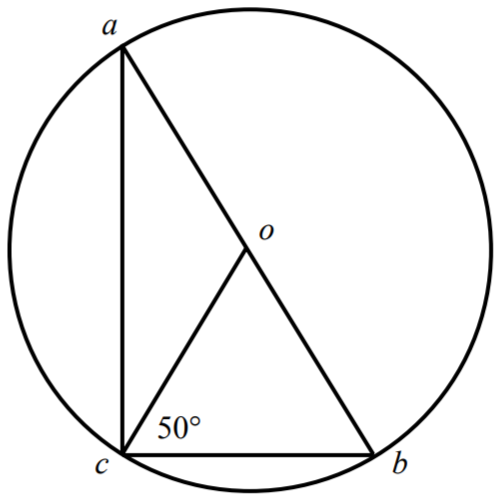
14.

$[ab]$ is the diameter of a circle of centre o.

$|\angle ocb| = 50°$.

(i) ✍ Find $|\angle boc|$.

(ii) ✍ Find $|\angle bac|$.



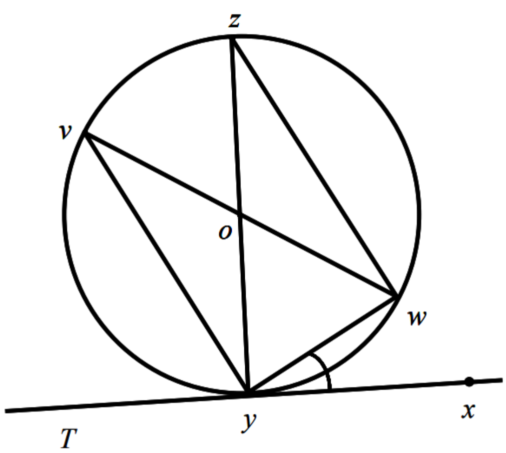
15.

T is a tangent to the circle and o is the centre of the circle.

$|\angle xyw| = 40°$.

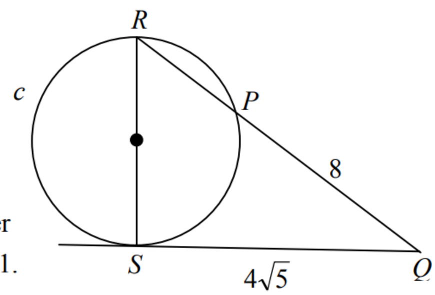
(i) ✍ Find $|\angle wvy|$.

(ii) ✍ Using congruent triangles or otherwise, prove $|zw| = |vy|$.



16. **Hint: Requires Pythagoras' Theorem**

The line QS is a tangent to the circle c.
[RS] is a diameter of the circle.
[QR] cuts the circle at P.
$|QP| = 8$ and $|QS| = 4\sqrt{5}$.



(i) Calculate $|RP|$.

(ii) Hence, calculate $|RS|$ and give your answer in the form $a\sqrt{b}$, where $a, b \in \mathbb{N}$ and $a > 1$.

17. Prove Theorems 9.37, 9.39 and 9.42.

9.13 Revision Solutions

1. 105°
2. (i) 100°
   (ii) 80°
3. (i) 55°
   (ii) 45°
4. (i) $x = 50^\circ$
   (ii) $y = 65^\circ$
5. (i) 55°
   (ii) 125°
6. 10.5
7. 15
8. 6
9. (i) $|pq| = |rs|$ (parallelogram), $|\angle qpt| = |\angle srw|$ (alternate) and $|\angle pqt| = |\angle wsr|$ (given) so triangles pqt, rsw are congruent (ASA) so that $|pt| = |wr$.
   (ii) If $|pt| = |wr$ then $|pw| = |tr|$. $|\angle spw| = |\angle prq|$ (alternate) and $|ps| = |qr|$ (parallelogram) making triangles psw, qtr congruent (SAS).
10. 10
11. (i) 90° (sits on diameter line)
    (ii) 45°
    (iii) $\sqrt{72}$
12. 75°
13. (i) 100°
    (ii) 130°
    (iii) 65°
14. (i) 80°
    (ii) 40°
15. (i) 40°
    (ii) $|\angle zyw| = 50^\circ$ (as $|\angle zyx| = 90^\circ$ as its radius meeting tangent). As $|oy| = |ow|$ (radius), $|\angle vwy| = |\angle zyw| = 50^\circ$ (isosceles). $|vw| = |yz|$ (diameter lines) and $|\angle wvy| = |\angle yzw|$ (standing on same arc), so triangles vwy, wyz are congruent. Therefore $|vy| = |wz|$.
16. (i) 2
    (ii) $2\sqrt{5}$