Projectiles Exam Question Solutions

Note These exam questions are given in reverse chronological order as they appear in exam papers; 2023 paper, Sample paper, 2022 (deferred), 2022, and so on back to 2015. Q3 (a) from 2005 is also included as an example of projectiles that bounce. They are answered in the style described in my notes. Only questions from the old syllabus relevant to the new syllabus are included.

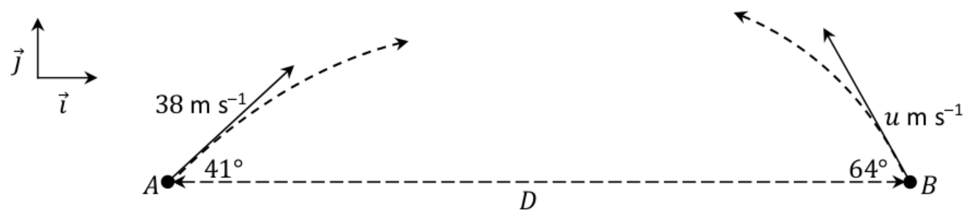
Question — 2023 Q8.

Question 8

Two balls, P and Q, are projected into the air from points A and B, which are a distance D apart along the horizontal $\vec{\imath}$ axis. The motion of the balls may be modelled as projectile motion in a vertical plane, ignoring the effects of air resistance.

P is projected from point A at time $t = 0$ s with initial velocity 38 m s^{-1} at 41° to AB.

Q is projected from point B at time $t = 1$ s with initial velocity u m s^{-1} at 64° to BA.



P and Q collide in mid-air when $t = 3$ s.

(i) Show that $u = 28$ m s^{-1} to the nearest whole number.

(ii) Calculate D.

(iii) In terms of $\vec{\imath}$ and $\vec{\jmath}$, calculate $\overrightarrow{v_P}$, the velocity of P, and $\overrightarrow{v_Q}$, the velocity of Q, when the balls collide, i.e. when $t = 3$ s.

(iv) Calculate the dot product of $\overrightarrow{v_P}$ and $\overrightarrow{v_Q}$ when $t = 3$ s.

(v) Hence or otherwise calculate the acute angle between $\overrightarrow{v_P}$ and $\overrightarrow{v_Q}$ when $t = 3$ s.

Using (u, θ):

(i) Calculating the velocity for P, partially for Q and setting $s_{yP} = s_{yQ} = s$ we have the following UVAST array for both objects.

P		Q	
x-axis	y-axis	x-axis	y-axis
$u_{xP} = 28.68$	$u_{yP} = 24.93$	$u_{xQ} = 0.44u$	$u_{yQ} = 0.9u$
$v_{xP} = 28.68$	$v_{yP} =$	$v_{xQ} = 0.44u$	$v_{yQ} =$
$a_{xP} = 0$	$a_{yP} = -g$	$a_{xQ} = 0$	$a_{yQ} = -g$
$s_{xP} =$	$s_{yP} = s$	$s_{xQ} =$	$s_{yQ} = s$
$t_{xP} = 3$	$t_{yP} = 3.$	$t_{xQ} = 2$	$t_{yQ} = 2.$

Then

$$\begin{aligned} s_{yP} &= u_{yP}t_{yP} + \frac{1}{2}a_{yP}t_{yP}^2 \\ \Rightarrow s &= 24.93(3) - 4.9(3)^2 \\ &= 30.69. \end{aligned}$$

$$\begin{aligned} s_{yQ} &= u_{yQ}t_{yQ} + \frac{1}{2}a_{yQ}t_{yQ}^2 \\ \Rightarrow 30.69 &= 0.93u(2) - 4.9(2)^2 \\ \Rightarrow 50.29 &= 1.8u \\ ' \Rightarrow 28 \text{ m/s} &= u. \end{aligned}$$

(ii)

$$\begin{aligned} D &= s_{xP} + s_{xQ} \\ &= u_{xP}t_{xP} + u_{xQ}t_{xQ} \\ &= 28.68(3) + 0.44(28)(2) \\ &= 110.68 \text{ m}. \end{aligned}$$

(iii)

$$\begin{aligned} v_{yP} &= u_{yP} + a_{yP}t_{yP} \\ &= 24.93 - 9.8(3) \\ &= -4.47 \end{aligned}$$

so that

$$\vec{v}_P = 28.68\,\vec{i} - 4.47\,\vec{j}.$$

$$\begin{aligned} v_{yQ} &= u_{yQ} + a_{yQ}t_{yQ} \\ &= 0.9(28) - 9.8(2) \\ &= 5.6 \end{aligned}$$

so that

$$\vec{v}_Q = 12.32\,\vec{i} + 5.6\,\vec{j}.$$

(iv) We can't actually have the positive $\vec{i}$ direction be different for P and Q when we apply the dot product. So

$$\vec{v}_P = 28.68\,\vec{i} - 4.47\,\vec{j},$$
$$\vec{v}_Q = -12.32\,\vec{i} + 5.6\,\vec{j}$$

so that

$$\begin{aligned}\vec{v}_P \cdot \vec{v}_Q &= 28.68(-12.32) - 4.47(5.6)\\ &= -378.3696\end{aligned}$$

(v) If θ is the angle between $\vec{v}_P$ and $\vec{v}_Q$ then

$$\begin{aligned}\cos\theta &= \frac{\vec{v}_P \cdot \vec{v}_Q}{|\vec{v}_P||\vec{v}_Q|}\\ &= \frac{-378.3696}{\sqrt{28.68^2 + 4.47^2}\sqrt{12.32^2 + 5.6^2}}\\ &= -0.96\\ \Rightarrow \theta &= 164^\circ.\end{aligned}$$

Therefore the acute angle between them is $180^\circ - 164^\circ = 16^\circ$.

Using (p,q)**:**

(i) Calculating the velocity for P, and $s_{yP} = s_{yQ} = s$ we have the following UVAST array for both objects.

P		Q	
x-axis	y-axis	x-axis	y-axis
$u_{xP} = 28.68$	$u_{yP} = 24.93$	$u_{xQ} = p$	$u_{yQ} = q$
$v_{xP} = 28.68$	$v_{yP} =$	$v_{xQ} = 0.44u$	$v_{yQ} =$
$a_{xP} = 0$	$a_{yP} = -g$	$a_{xQ} = 0$	$a_{yQ} = -g$
$s_{xP} =$	$s_{yP} = s$	$s_{xQ} =$	$s_{yQ} = s$
$t_{xP} = 3$	$t_{yP} = 3.$	$t_{xQ} = 2$	$t_{yQ} = 2.$

Then

$$\begin{aligned}s_{yP} &= u_{yP}t_{yP} + \frac{1}{2}a_{yP}t_{yP}^2\\ \Rightarrow s &= 24.93(3) - 4.9(3)^2\\ &= 30.69.\end{aligned}$$

$$
\begin{aligned}
s_{yQ} &= u_{yQ}t_{yQ} + \frac{1}{2}a_{yQ}t_{yQ}^2 \\
\Rightarrow 30.69 &= q(2) - 4.9(2)^2 \\
\Rightarrow 50.29 &= 2q \\
' \Rightarrow 25.15 &= q.
\end{aligned}
$$

Then

$$
\begin{aligned}
\tan 64^\circ &= \frac{q}{p} \\
\Rightarrow p\tan 64^\circ &= 25.15 \\
\Rightarrow p &= 12.23.
\end{aligned}
$$

Then

$$
\begin{aligned}
u &= \sqrt{p^2+q^2} \\
&= 28 \text{ m/s}.
\end{aligned}
$$

(ii) Same as with u, θ.

(iii) Same as with u, θ.

(iv) Same as with u, θ.

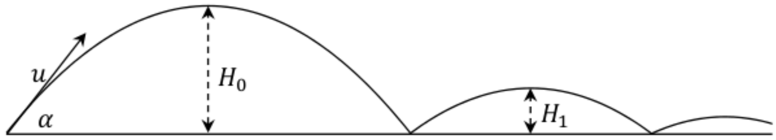
(v) Same as with u, θ.

Question — Sample Q4 (a).

Question 4

(a) A ball is projected from a point on horizontal ground, with initial speed u and at an angle α to the horizontal. The ball reaches a maximum height of H_0 above the horizontal.

Upon landing, the ball bounces with a maximum height of H_1.



The coefficient of restitution between the ball and the ground is e.

(i) Calculate $\frac{H_0}{H_1}$.

(ii) The ball continues bouncing. Find an expression (in terms of e and H_0) for H_5, the maximum height of the ball after it lands on the ground for the fifth time.

Using (u, θ):

(i) First, we can find H_0 easily enough.

x-axis	y-axis
$u_x = u\cos\alpha$	$u_y = u\sin\alpha$
$v_x = u\cos\alpha$	$v_y = 0$
$a_x = 0$	$a_y = -g$
$s_x =$	$s_y = H_0$
$t_x =$	$t_y = .$

$$v_y^2 = u_y^2 + 2a_y s_y$$
$$\Rightarrow 0 = u^2\sin^2\alpha - 2gH_0$$
$$\Rightarrow 2gH_0 = u^2\sin^2\alpha$$
$$\Rightarrow H_0 = \frac{u^2\sin^2\alpha}{2g}.$$

We have the following UVAST array representing the two journeys, ending when the object is at height H_1.

First Part		Second Part		Extra Equations
x-axis	y-axis	x-axis	y-axis	
$u_{x1} = u\cos\alpha$	$u_{y1} = u\sin\alpha$	$u_{x2} = u\cos\alpha$	$u_{y2} =$	
$v_{x1} = u\cos\alpha$	$v_{y1} =$	$v_{x2} = u\cos\alpha$	$v_{y2} = 0$	
$a_{x1} = 0$	$a_{y1} = -g$	$a_{x2} = 0$	$a_{y2} = -g$	$u_{y2} = -ev_{y1}$
$s_{x1} =$	$s_{y1} = 0$	$s_{x2} =$	$s_{y2} = H_1$	
$t_{x1} = t_1$	$t_{y1} = t_1$	$t_{x2} = t_2$	$t_{y2} = t_2$	

First,

$$\begin{aligned} s_{y1} &= \left(\frac{u_{y1}+v_{y1}}{2}\right)t_{y1} \\ \Rightarrow 0 &= \left(\frac{u\sin\theta+v_{y1}}{2}\right)t_1 \\ \Rightarrow 0 &= u\sin\theta + v_{y1} \\ \Rightarrow -u\sin\theta &= v_{y1}. \end{aligned}$$

Then

$$\begin{aligned} u_{y2} &= -ev_{y1} \\ &= eu\sin\theta. \end{aligned}$$

Finally

$$\begin{aligned} v_{y2}^2 &= u_{y2}^2 + a_{y2}s_{y2} \\ \Rightarrow 0 &= e^2u^2\sin^2\alpha - 2gH_1 \\ \Rightarrow 2gH_1 &= e^2u^2\sin^2\alpha \\ \Rightarrow H_1 &= \frac{e^2u^2\sin^2\alpha}{2g}. \end{aligned}$$

Therefore

$$\begin{aligned} \frac{H_1}{H_0} &= \frac{\frac{e^2u^2\sin^2\alpha}{2g}}{\frac{u^2\sin^2\alpha}{2g}} \\ &= e^2. \end{aligned}$$

(ii) H_n is a geometric sequence with common ratio e^2 and first term ($n = 1$ term) $H_1 = e^2H_0$. Therefore

$$\begin{aligned} H_5 &= e^2H_0(e^2)^4 \\ &= e^{10}H_0. \end{aligned}$$

Using (p,q):

(i) First, we can find H_0 easily enough.

x-axis	y-axis
$u_x = p$	$u_y = q$
$v_x = p$	$v_y = 0$
$a_x = 0$	$a_y = -g$
$s_x =$	$s_y = H_0$
$t_x =$	$t_y = .$

$$\begin{aligned} v_y^2 &= u_y^2 + 2a_y s_y \\ \Rightarrow 0 &= q^2 - 2gH_0 \\ \Rightarrow 2gH_0 &= q^2 \\ \Rightarrow H_0 &= \frac{q^2}{2g}. \end{aligned}$$

We have the following UVAST array representing the two journeys, ending when the object is at height H_1.

First Part		Second Part		Extra Equations
x-axis	y-axis	x-axis	y-axis	
$u_{x1} = p$	$u_{y1} = q$	$u_{x2} = p$	$u_{y2} =$	
$v_{x1} = p$	$v_{y1} =$	$v_{x2} = p$	$v_{y2} = 0$	$u_{y2} = -ev_{y1}$
$a_{x1} = 0$	$a_{y1} = -g$	$a_{x2} = 0$	$a_{y2} = -g$	
$s_{x1} =$	$s_{y1} = 0$	$s_{x2} =$	$s_{y2} = H_1$	
$t_{x1} = t_1$	$t_{y1} = t_1$	$t_{x2} = t_2$	$t_{y2} = t_2$	

First,

$$\begin{aligned} s_{y1} &= \left(\frac{u_{y1} + v_{y1}}{2}\right) t_{y1} \\ \Rightarrow 0 &= \left(\frac{q + v_{y1}}{2}\right) t_1 \\ \Rightarrow 0 &= q + v_{y1} \\ \Rightarrow -q &= v_{y1}. \end{aligned}$$

Then

$$\begin{aligned} u_{y2} &= -ev_{y1} \\ &= eq. \end{aligned}$$

Finally

$$\begin{aligned} v_{y2}^2 &= u_{y2}^2 + a_{y2} s_{y2} \\ \Rightarrow 0 &= e^2 q^2 - 2gH_1 \\ \Rightarrow 2gH_1 &= e^2 q^2 \\ \Rightarrow H_1 &= \frac{e^2 q^2}{2g}. \end{aligned}$$

Therefore

$$\begin{aligned} \frac{H_1}{H_0} &= \frac{\frac{e^2 q^2}{2g}}{\frac{q^2}{2g}} \\ &= e^2. \end{aligned}$$

(ii) H_n is a geometric sequence with common ratio e^2 and first term ($n = 1$ term) $H_1 = e^2 H_0$. Therefore

$$\begin{aligned} H_5 &= e^2 H_0 (e^2)^4 \\ &= e^{10} H_0. \end{aligned}$$

Question — 2022 (Deferred) Q3 (a).

(a) A particle is projected from a point on horizontal ground. The speed of projection is 14 m s^{-1} at an angle α to the horizontal.

Find the two values of α that will give a range of 10 m.

Using (u, θ):

We have the following UVAST array.

x-axis	y-axis
$u_x = 14\cos\alpha$	$u_y = 14\sin\alpha$
$v_x = 14\cos\alpha$	$v_y =$
$a_x = 0$	$a_y = -g$
$s_x = 10$	$s_y = 0$
$t_x = t$	$t_y = t$.

As being given the range is just a special (and algebraically easier) case of being given a target hit, we proceed as in those problems.

$$\begin{aligned} s_x &= u_x t_x \\ \Rightarrow 10 &= 14\cos\alpha t \\ \Rightarrow \frac{5}{7\cos\alpha} &= t. \end{aligned}$$

Then

$$\begin{aligned} s_y &= u_y t_y + \frac{1}{2}a_y t_y^2 \\ \Rightarrow 0 &= 14\sin\alpha\left(\frac{5}{7\cos\alpha}\right) - 4.9\left(\frac{5}{7\cos\alpha}\right)^2 \\ \Rightarrow 0 &= 10\frac{\sin\alpha}{\cos\alpha} - 4.9\left(\frac{25}{49\cos^2\alpha}\right) \\ \Rightarrow 0 &= 10\tan\alpha - 2.5\left(1+\tan^2\alpha\right) \end{aligned}$$

$$\begin{aligned} \Rightarrow 2.5\tan^2\alpha - 10\tan\alpha + 2.5 &= 0 \\ \Rightarrow \tan^2\alpha - 4\tan\alpha + 1 &= 0 \end{aligned}$$

$$\begin{aligned} \Rightarrow \tan\alpha &= \frac{4 \pm \sqrt{4^2 - 4(1)(1)}}{2(1)} \\ &= 0.27, 3.73 \\ \Rightarrow \alpha &= 15^\circ, 75^\circ. \end{aligned}$$

Using (p, q):

First we know that

$$\sqrt{p^2+q^2} = 14$$
$$\Rightarrow p^2+q^2 = 196.$$

We have the following UVAST array.

$$\begin{array}{ll} \underline{x\text{-axis}} & \underline{y\text{-axis}} \\ u_x = p & u_y = q \\ v_x = p & v_y = \\ a_x = 0 & a_y = -g \\ s_x = 10 & s_y = 0 \\ t_x = t & t_y = t. \end{array}$$

As being given the range is just a special (and algebraically easier) case of being given a target hit, we proceed as in those problems.

$$\begin{aligned} s_x &= u_x t_x \\ \Rightarrow 10 &= pt \\ \Rightarrow \frac{10}{p} &= t. \end{aligned}$$

Then

$$\begin{aligned} s_y &= u_y t_y + \frac{1}{2} a_y t_y^2 \\ \Rightarrow 0 &= q\left(\frac{10}{p}\right) - \frac{g}{2}\left(\frac{10}{p}\right)^2 \\ \Rightarrow 0 &= \frac{10q}{p} - \frac{g}{2}\left(\frac{100}{p^2}\right) \\ \Rightarrow \frac{50g}{p^2} &= \frac{10q}{p} \\ \Rightarrow \frac{49}{p} &= q. \end{aligned}$$

Plugging this into our other equation yields

$$
\begin{aligned}
p^2 + \left(\frac{49}{p^2}\right) &= 196 \\
\Rightarrow p^2 + \frac{2401}{p^2} &= 196 \\
\Rightarrow p^4 + 2401 &= 196p^2 \\
\Rightarrow p^4 - 196p^2 + 2401 &= 0 \\
\Rightarrow p^2 &= \frac{196 \pm \sqrt{196^2 - 4(2401)}}{2} \\
&= 13.13, 182.87 \\
\Rightarrow p &= 3.62, 13.52 \\
\Rightarrow q &= \frac{49}{3.62}, \frac{49}{13.52} \\
&= 13.54, 3.62.
\end{aligned}
$$

We still need to find α.

$$
\begin{aligned}
\tan\alpha &= \frac{q}{p} \\
&= 3.74, 0.27 \\
\Rightarrow \alpha &= 75^\circ, 15^\circ.
\end{aligned}
$$

Question — 2022 Q3 (a).

(a) A particle is projected out to sea from a point P on a cliff to hit a target 60 m horizontally from P and 60 m vertically below P.

The velocity of projection is $14\sqrt{3}$ m s^{-1} at an angle α to the horizontal.

Find

(i) the two possible values of α

(ii) the times of flight.

Using (u, θ):

(i) We have the following UVAST array. Note s_y is negative as we strike the target below P.

x-axis	y-axis
$u_x = 14\sqrt{3}\cos\alpha$	$u_y = 14\sqrt{3}\sin\alpha$
$v_x = 14\sqrt{3}\cos\alpha$	$v_y =$
$a_x = 0$	$a_y = -g$
$s_x = 60$	$s_y = -60$
$t_x = t$	$t_y = t$.

This is a standard target practice problem ($s_y < 0$ doesn't make the algebra any harder) and so we proceed as in those problems.

$$\begin{aligned} s_x &= u_x t_x \\ \Rightarrow 60 &= 14\sqrt{3}\cos\alpha t \\ \Rightarrow \frac{30}{7\sqrt{3}\cos\alpha} &= t. \end{aligned}$$

Then

$$\begin{aligned} s_y &= u_y t_y + \frac{1}{2}a_y t_y^2 \\ \Rightarrow -60 &= 14\sqrt{3}\sin\alpha\left(\frac{30}{7\sqrt{3}\cos\alpha}\right) - 4.9\left(\frac{30}{7\sqrt{3}\cos\alpha}\right)^2 \\ \Rightarrow -60 &= 60\frac{\sin\alpha}{\cos\alpha} - 4.9\left(\frac{900}{49(3)\cos^2\alpha}\right) \\ \Rightarrow -60 &= 60\tan\alpha - 30\left(1+\tan^2\alpha\right) \\ \Rightarrow 30\tan^2\alpha - 60\tan\alpha - 30 &= 0 \\ \Rightarrow \tan^2\alpha - 2\tan\alpha - 1 &= 0 \\ \Rightarrow \tan\alpha &= \frac{2 \pm \sqrt{2^2 - 4(1)(-1)}}{2(1)} \\ &= 1 \pm \sqrt{2} \\ \Rightarrow \alpha &= 67.5^\circ, -22.5^\circ. \end{aligned}$$

(ii)

$$\begin{aligned} t &= \frac{30}{7\sqrt{3}\cos 67.5^\circ}, \frac{30}{7\sqrt{3}\cos -22.5^\circ} \\ &= 6.47, 2.68 \text{ seconds.} \end{aligned}$$

Using (p,q)**:**

(i) First we know that

$$\begin{aligned} \sqrt{p^2+q^2} &= 14\sqrt{3} \\ \Rightarrow p^2+q^2 &= 588. \end{aligned}$$

We have the following UVAST array.

x-axis	y-axis
$u_x = p$	$u_y = q$
$v_x = p$	$v_y =$
$a_x = 0$	$a_y = -g$
$s_x = 60$	$s_y = -60$
$t_x = t$	$t_y = t.$

This is a standard target practice problem ($s_y < 0$ doesn't make the algebra any harder) and so we proceed as in those problems.

$$\begin{aligned} s_x &= u_x t_x \\ \Rightarrow 60 &= pt \\ \Rightarrow \frac{60}{p} &= t. \end{aligned}$$

Then

$$\begin{aligned} s_y &= u_y t_y + \frac{1}{2} a_y t_y^2 \\ \Rightarrow -60 &= q\left(\frac{60}{p}\right) - \frac{g}{2}\left(\frac{60}{p}\right)^2 \\ \Rightarrow -60 &= \frac{60q}{p} - \frac{g}{2}\left(\frac{3600}{p^2}\right) \\ \Rightarrow \frac{17640}{p^2} - 60 &= \frac{60q}{p} \\ \Rightarrow \frac{294}{p} - p &= q. \end{aligned}$$

Plugging this into our other equation yields

$$\begin{aligned}
p^2 + \left(\frac{294}{p} - p\right)^2 &= 588 \\
\Rightarrow p^2 + \frac{86436}{p^2} - 588 + p^2 &= 588 \\
\Rightarrow 2p^2 - 1176 + \frac{86436}{p^2} &= 0 \\
\Rightarrow 2p^4 - 1176p^2 + 86436 &= 0 \\
\Rightarrow p^4 - 588p^2 + 43218 &= 0
\end{aligned}$$

$$\begin{aligned}
\Rightarrow p^2 &= \frac{588 \pm \sqrt{588^2 - 4(43218)}}{2} \\
&= 294 \pm 147\sqrt{2} \\
\Rightarrow p &= 9.28, 22.4 \\
\Rightarrow q &= \frac{294}{9.28} - 9.28, \frac{294}{22.4} - 22.4 \\
&= 22.4, -9.275.
\end{aligned}$$

We still need to find α.

$$\begin{aligned}
\tan\alpha &= \frac{q}{p} \\
&= 2.41, -0.41 \\
\Rightarrow \alpha &= 67^\circ, -22^\circ.
\end{aligned}$$

(ii)

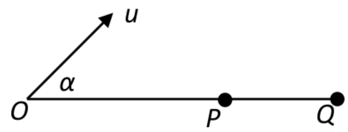
$$\begin{aligned}
t &= \frac{60}{9.28}, \frac{60}{22.4} \\
&= 6.47, 2.68 \text{ seconds.}
\end{aligned}$$

Question — 2021 Q3 (a).

(a) A particle is projected from a point O with speed u m s^{-1} at an angle α to the horizontal.

(i) Show that the range of the particle is $\frac{u^2 \sin 2\alpha}{g}$,

and that the maximum range $|OQ|$ is $\frac{u^2}{g}$.



If the angle of projection is increased to 60° the particle strikes the horizontal plane at P.

(ii) Find the distance $|PQ|$ in terms of u.

Using (u, θ):

(i) We have the following UVAST array.

x-axis	y-axis
$u_x = u\cos\alpha$	$u_y = u\sin\alpha$
$v_x = u\cos\alpha$	$v_y =$
$a_x = 0$	$a_y = -g$
$s_x =$	$s_y = 0$
$t_x = t$	$t_y = t.$

$$\begin{aligned} s_y &= u_y t_y + \frac{1}{2}a_y t_y^2 \\ \Rightarrow 0 &= u\sin\alpha t - \frac{g}{2}t^2 \\ \Rightarrow \frac{g}{2}t^2 &= u\sin\alpha t \\ \Rightarrow t &= \frac{2u\sin\alpha}{g}. \end{aligned}$$

Then

$$\begin{aligned} s_x &= u_x t_x \\ &= u\cos\alpha \frac{2u\sin\alpha}{g} \\ &= \frac{2u^2\cos\alpha\sin\alpha}{g} \\ &= \frac{u^2\sin 2\alpha}{g}. \end{aligned}$$

As $\sin 2\alpha \leq 1$ for any α and is equal to 1 when $\alpha = 45°$, the maximum range is

$$\frac{u^2}{g}.$$

(ii) From part (i), when $\alpha = 60^\circ$ the range is

$$\frac{u^2 \sin 2(60^\circ)}{g} = \frac{u^2\sqrt{3}}{2g}.$$

Using (p,q):

(i) We have the following UVAST array.

$$\begin{array}{ll} \underline{x\text{-axis}} & \underline{y\text{-axis}} \\ u_x = p & u_y = q \\ v_x = q & v_y = \\ a_x = 0 & a_y = -g \\ s_x = & s_y = 0 \\ t_x = t & t_y = t. \end{array}$$

$$\begin{aligned} s_y &= u_y t_y + \frac{1}{2} a_y t_y^2 \\ \Rightarrow 0 &= qt - \frac{g}{2}t^2 \\ \Rightarrow \frac{g}{2}t^2 &= qt \\ \Rightarrow t &= \frac{2q}{g}. \end{aligned}$$

Then

$$\begin{aligned} s_x &= u_x t_x \\ &= p\frac{2q}{g} \\ &= \frac{2pq}{g}. \end{aligned}$$

Even though we used p,q we still need to convert our answer into an expression in (u,θ) in the end.

$$\begin{aligned} s_x &= \frac{2(u\cos\theta)(u\sin\theta)}{g} \\ &= \frac{2u^2\cos\alpha\sin\alpha}{g} \\ &= \frac{u^2\sin 2\alpha}{g}. \end{aligned}$$

As $\sin 2\alpha \leq 1$ for any α and is equal to 1 when $\alpha = 45^\circ$, the maximum range is

$$\frac{u^2}{g}.$$

(ii) From part (i), when $\alpha = 60^\circ$ the range is

$$\frac{u^2 \sin 2(60^\circ)}{g} = \frac{u^2\sqrt{3}}{2g}.$$

Question — 2020 Q3 (a).

(a) A particle is projected from a point P with speed u m s^{-1} at an angle α to the horizontal.

(i) Show that the range of the particle is $\frac{2u^2 \sin\alpha \cos\alpha}{g}$.

The particle is 24·5 m above the horizontal ground after 5 seconds and it strikes the ground 235·2 m from P.

(ii) Find the value of u.

Using (u, θ):

(i) We have the following UVAST array.

$$\begin{array}{ll} \underline{x\text{-axis}} & \underline{y\text{-axis}} \\ u_x = u\cos\alpha & u_y = u\sin\alpha \\ v_x = u\cos\alpha & v_y = \\ a_x = 0 & a_y = -g \\ s_x = & s_y = 0 \\ t_x = t & t_y = t. \end{array}$$

$$\begin{aligned} s_y &= u_y t_y + \frac{1}{2} a_y t_y^2 \\ \Rightarrow 0 &= u\sin\alpha t - \frac{g}{2}t^2 \\ \Rightarrow \frac{g}{2}t^2 &= u\sin\alpha t \\ \Rightarrow t &= \frac{2u\sin\alpha}{g}. \end{aligned}$$

Then

$$\begin{aligned} s_x &= u_x t_x \\ &= u\cos\alpha \frac{2u\sin\alpha}{g} \\ &= \frac{2u^2\cos\alpha\sin\alpha}{g}. \end{aligned}$$

(ii) The second piece of information tells us that

$$\frac{2u^2\cos\alpha\sin\alpha}{g} = 235.2.$$

The first gives us the following UVAST array.

x-axis	y-axis
$u_x = u\cos\alpha$	$u_y = u\sin\alpha$
$v_x = u\cos\alpha$	$v_y =$
$a_x = 0$	$a_y = -g$
$s_x =$	$s_y = 24.5$
$t_x = 5$	$t_y = 5.$

$$s_y = u_y t_y + \frac{1}{2}a_y t_y^2$$
$$\Rightarrow 24.5 = 5u\sin\alpha - 122.5$$
$$\Rightarrow 147 = 5u\sin\alpha.$$

Given the two simultaneous equations

$$\frac{2u^2\cos\alpha\sin\alpha}{g} = 235.2,$$
$$5u\sin\alpha = 147$$

it's easiest to write the second one in terms of u and substitute it into the first one.

$$5u\sin\alpha = 147$$
$$\Rightarrow u = \frac{29.4}{\sin\alpha}$$
$$\Rightarrow \frac{2\left(\frac{29.4}{\sin\alpha}\right)^2\cos\alpha\sin\alpha}{g} = 235.2$$
$$\Rightarrow \frac{2\left(\frac{864.36}{\sin^2\alpha}\right)\cos\alpha\sin\alpha}{g} = 235.2$$
$$\Rightarrow 176.4\frac{\cos\alpha}{\alpha\alpha} = 235.2$$
$$\Rightarrow \frac{1}{\tan\alpha} = \frac{4}{3}$$
$$\Rightarrow \tan\alpha = \frac{3}{4}$$
$$\Rightarrow \sin\alpha = \frac{3}{5} \quad \text{(by drawing a triangle or otherwise)}$$
$$\Rightarrow u = \frac{29.4}{\frac{3}{5}}$$
$$= 49 \text{ m/s.}$$

Using (p,q):

(i) We have the following UVAST array.

$$\begin{array}{ll} \underline{x\text{-axis}} & \underline{y\text{-axis}} \\ u_x = p & u_y = q \\ v_x = q & v_y = \\ a_x = 0 & a_y = -g \\ s_x = & s_y = 0 \\ t_x = t & t_y = t. \end{array}$$

$$\begin{aligned} s_y &= u_y t_y + \frac{1}{2} a_y t_y^2 \\ \Rightarrow 0 &= qt - \frac{g}{2} t^2 \\ \Rightarrow \frac{g}{2} t^2 &= qt \\ \Rightarrow t &= \frac{2q}{g}. \end{aligned}$$

Then

$$\begin{aligned} s_x &= u_x t_x \\ &= p \frac{2q}{g} \\ &= \frac{2pq}{g}. \end{aligned}$$

Even though we used p, q we still need to convert our answer into an expression in (u, θ) in the end.

$$\begin{aligned} s_x &= \frac{2(u \cos\theta)(u \sin\theta)}{g} \\ &= \frac{2u^2 \cos\alpha \sin\alpha}{g}. \end{aligned}$$

(ii) The second piece of information tells us that

$$\frac{2pq}{g} = 235.2.$$

The first gives us the following UVAST array.

$$\begin{array}{ll} \underline{x\text{-axis}} & \underline{y\text{-axis}} \\ u_x = p & u_y = q \\ v_x = p & v_y = \\ a_x = 0 & a_y = -g \\ s_x = & s_y = 24.5 \\ t_x = 5 & t_y = 5. \end{array}$$

$$
\begin{aligned}
s_y &= u_y t_y + \frac{1}{2} a_y t_y^2 \\
\Rightarrow 24.5 &= 5q - 122.5 \\
\Rightarrow 29.4 &= q.
\end{aligned}
$$

Then, going back to the first equation,

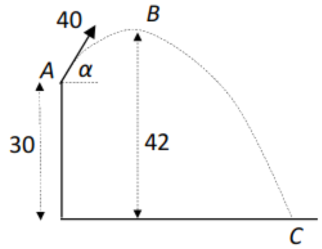
$$
\begin{aligned}
\frac{2pq}{g} &= 235.2 \\
\Rightarrow \frac{2p(29.4)}{g} &= 235.2 \\
\Rightarrow p &= 39.2.
\end{aligned}
$$

Finally, we still have to find u.

$$
\begin{aligned}
u &= \sqrt{p^2 + q^2} \\
&= \sqrt{39.2^2 + 29.4^2} \\
&= 49 \text{ m/s}.
\end{aligned}
$$

Question — 2019 Q3 (a).

3. (a) A particle is projected with speed 40 m s^{-1} from a point A on the top of a vertical cliff of height 30 m.
The maximum height reached by the particle is 42 m above the horizontal ground, at point B.
It strikes the ground at C.



Find

(i) the value of α, the angle of projection

(ii) the horizontal range of the particle

(iii) the speed of the particle as it hits the ground at C.

Using (u,θ)**:**

(i) Ending the journey at B we have the following UAVST array. We have the following UVAST array.

x-axis	y-axis
$u_x = 40\cos\alpha$	$u_y = 40\sin\alpha$
$v_x = 40\cos\alpha$	$v_y = 0$
$a_x = 0$	$a_y = -g$
$s_x =$	$s_y = 12$
$t_x = t$	$t_y = t.$

$$\begin{aligned} v_y^2 &= u_y^2 + 2a_y s_y \\ \Rightarrow 0^2 &= (40\sin\alpha)^2 - 2g(12) \\ \Rightarrow 24g &= 1600\sin^2\alpha \\ \Rightarrow 0.147 &= \sin^2\alpha \\ \Rightarrow \sqrt{0.147} &= \sin\alpha \\ \Rightarrow 22.5 \approx &= \alpha. \end{aligned}$$

(ii) Drawing a new UVAST array where the journey ends at C, with α now known.

x-axis	y-axis
$u_x = 36.94$	$u_y = 15.34$
$v_x = 36.94$	$v_y =$
$a_x = 0$	$a_y = -g$
$s_x =$	$s_y = -30$
$t_x = t$	$t_y = t.$

$$
\begin{aligned}
s_y &= u_y t_y + \frac{1}{2} a_y t_y^2 \\
\Rightarrow -30 &= 15.34t - 4.9t^2 \\
\Rightarrow 4.9t^2 - 15.34t - 30 &= 0 \\
\Rightarrow t &= \frac{15.34 \pm \sqrt{15.34^2 - 4(4.9)(-30)}}{2(4.9)} \\
&= 4.49, \cancel{-1.36} \\
\Rightarrow s_x &= u_x t_x \\
&= 36.94(4.49) \\
&= 165.88 \text{ m.}
\end{aligned}
$$

(iii) It's actually easiest to use the Principle of Conservation of Energy to calculate the speed (unlike if we were asked for velocity). Letting the ground under the cliff be the baseline when calculating potential energy heights and letting v be the speed at C,

$$
\begin{aligned}
\text{P.E.}_1 + \text{K.E.}_1 &= \text{P.E.}_2 + \text{K.E.}_2 \\
\Rightarrow mg(30) + \frac{m(40)^2}{2} &= 0 + \frac{mv^2}{2} \\
\Rightarrow 60g + 1600 &= v^2 \\
\Rightarrow 46.78 \text{ m/s} &= v.
\end{aligned}
$$

Using (p,q):

(i) First note that

$$\sqrt{p^2+q^2} = 40.$$

Ending the journey at B we have the following UAVST array. We have the following UVAST array.

x-axis	y-axis
$u_x = p$	$u_y = q$
$v_x = p$	$v_y = 0$
$a_x = 0$	$a_y = -g$
$s_x =$	$s_y = 12$
$t_x = t$	$t_y = t.$

$$
\begin{aligned}
v_y^2 &= u_y^2 + 2a_y s_y \\
\Rightarrow 0^2 &= q^2 - 2g(12) \\
\Rightarrow 24g &= q^2 \\
\Rightarrow \sqrt{24g} &= q.
\end{aligned}
$$

Then

$$\begin{aligned} \sqrt{p^2+q^2} &= 40 \\ \Rightarrow p^2+24g &= 40^2 \\ \Rightarrow p &\approx 36.94 \\ \Rightarrow \alpha &= \tan^{-1}\left(\frac{q}{p}\right) \\ &\approx 22.5^\circ. \end{aligned}$$

(ii) Drawing a new UVAST array where the journey ends at C, with p and q now known.

x-axis	y-axis
$u_x = 36.94$	$u_y = 15.34$
$v_x = 36.94$	$v_y =$
$a_x = 0$	$a_y = -g$
$s_x =$	$s_y = -30$
$t_x = t$	$t_y = t.$

$$\begin{aligned} s_y &= u_y t_y + \frac{1}{2}a_y t_y^2 \\ \Rightarrow -30 &= 15.34t - 4.9t^2 \\ \Rightarrow 4.9t^2 - 15.34t - 30 &= 0 \\ \Rightarrow t &= \frac{15.34 \pm \sqrt{15.34^2 - 4(4.9)(-30)}}{2(4.9)} \\ &= 4.49, \cancel{-1.36} \\ \Rightarrow s_x &= u_x t_x \\ &= 36.94(4.49) \\ &= 165.88 \text{ m.} \end{aligned}$$

(iii) It's actually easiest to use the Principle of Conservation of Energy to calculate the speed (unlike if we were asked for velocity). Letting the ground under the cliff be the baseline when calculating potential energy heights and letting v be the speed at C,

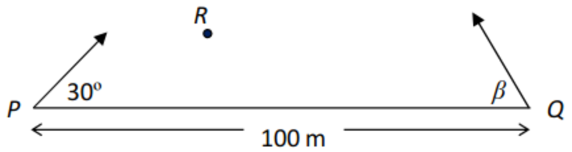
$$\begin{aligned} \text{P.E.}_1 + \text{K.E.}_1 &= \text{P.E.}_2 + \text{K.E.}_2 \\ \Rightarrow mg(30) + \frac{m(40)^2}{2} &= 0 + \frac{mv^2}{2} \\ \Rightarrow 60g + 1600 &= v^2 \\ \Rightarrow 46.78 \text{ m/s} &= v. \end{aligned}$$

Question — 2018 Q3 (a).

3. (a) A particle is projected from a point *P* with speed 60 m s^{-1} at an angle of 30° to the horizontal. At the same time a second particle is projected from a point *Q* with speed 50 m s^{-1} at an angle β to the horizontal. *P* and *Q* are on the same horizontal level and are 100 m apart. The particles collide at *R* as shown in the diagram.

(i) Show that $\sin\beta = \frac{3}{5}$.

(ii) Find the distance $|PR|$.



Using (u, θ):

(i) Calculating the velocity for P, partially for Q and setting $s_{yP} = s_{yQ} = s$ as well as all times equal we have the following UVAST array for both objects.

P: x-axis	P: y-axis	Q: x-axis	Q: y-axis
$u_{xP} = 30\sqrt{3}$	$u_{yP} = 30$	$u_{xQ} = 50\cos\beta$	$u_{yQ} = 50\sin\beta$
$v_{xP} = 30\sqrt{3}$	$v_{yP} =$	$v_{xQ} = 50\cos\beta$	$v_{yQ} =$
$a_{xP} = 0$	$a_{yP} = -g$	$a_{xQ} = 0$	$a_{yQ} = -g$
$s_{xP} =$	$s_{yP} = s$	$s_{xQ} =$	$s_{yQ} = s$
$t_{xP} = t$	$t_{yP} = t$	$t_{xQ} = t$	$t_{yQ} = t.$

Using $s = ut + \frac{1}{2}at^2$ on both y columns and setting them equal we have

$$\begin{aligned} u_{yP}t_{yP} + \frac{1}{2}a_{yP}t_{yP}^2 &= u_{yQ}t_{yQ} + \frac{1}{2}a_{yQ}t_{yQ}^2 \\ \Rightarrow 30t - \frac{g}{2}t^2 &= 50\sin\beta t - \frac{g}{2}t^2 \\ \Rightarrow 30t &= 50\sin\beta t \\ \Rightarrow \frac{3}{5} &= \sin\beta. \end{aligned}$$

(ii) By drawing a triangle or otherwise we can quickly show that $\cos\beta = \frac{4}{5}$. Then

$$\begin{aligned} |PQ| &= s_{xP} + s_{yP} \\ \Rightarrow 100 &= u_{xP}t_{xP} + u_{xQ}t_{xQ} \\ \Rightarrow 100 &= 30\sqrt{3}t + 40t \\ \Rightarrow 1.09 &= t. \end{aligned}$$

Then

$$\begin{aligned} s_{xP} &= 30\sqrt{3}t \\ &= 56.5 \end{aligned}$$

and

$$\begin{aligned} s_{yP} &= u_{yP}t_{yP} + \frac{1}{2}a_{yP}t_{yP}^2 \\ &= 30\,(1.09) - \frac{g}{2}(1.09)^2 \\ &26.88. \end{aligned}$$

Then

$$\begin{aligned} |PR| &= \sqrt{s_{xP}^2 + s_{yP}^2} \\ &= 62.57 \text{ m.} \end{aligned}$$

Using (p,q):

(i) Calculating the velocity for P, partially for Q and setting $s_{yP} = s_{yQ} = s$ as well as all times equal we have the following UVAST array for both objects.

P		Q	
x-axis	y-axis	x-axis	y-axis
$u_{xP} = 30\sqrt{3}$	$u_{yP} = 30$	$u_{xQ} = p$	$u_{yQ} = q$
$v_{xP} = 30\sqrt{3}$	$v_{yP} =$	$v_{xQ} = p$	$v_{yQ} =$
$a_{xP} = 0$	$a_{yP} = -g$	$a_{xQ} = 0$	$a_{yQ} = -g$
$s_{xP} =$	$s_{yP} = s$	$s_{xQ} =$	$s_{yQ} = s$
$t_{xP} = t$	$t_{yP} = t$	$t_{xQ} = t$	$t_{yQ} = t.$

Using $s = ut + \frac{1}{2}at^2$ on both y columns and setting them equal we have

$$\begin{aligned} u_{yP}t_{yP} + \frac{1}{2}a_{yP}t_{yP}^2 &= u_{yQ}t_{yQ} + \frac{1}{2}a_{yQ}t_{yQ}^2 \\ \Rightarrow 30t - \frac{g}{2}t^2 &= qt - \frac{g}{2}t^2 \\ \Rightarrow 30t &= qt \\ \Rightarrow 30 &= q. \end{aligned}$$

Then

$$\begin{aligned} \sqrt{p^2 + q^2} &= 50 \\ \Rightarrow p^2 + 900 &= 2500 \\ \Rightarrow p &= 40 \\ \Rightarrow \tan\beta &= \frac{q}{p} \\ &= \frac{4}{3}. \end{aligned}$$

By drawing a triangle or otherwise we can quickly show that $\sin\beta = \frac{4}{5}$.

(ii) Same as with u, θ.

Question — 2017 Q3 (a).

3. (a) A particle is projected with speed $\sqrt{\frac{9gh}{2}}$ from a point P on the top of a cliff of height h. It strikes the ground a horizontal distance $3h$ from P.

(i) Find the two possible angles of projection.

(ii) For each angle of projection find, in terms of h, the time it takes the particle to reach P.

Note: (ii) should read "hits the ground", not "reaches P".

Using (u, θ):

(i) We have the following UVAST array.

x-axis	y-axis
$u_x = \sqrt{\frac{9gh}{2}}\cos\theta$	$u_y = \sqrt{\frac{9gh}{2}}\sin\theta$
$v_x = \sqrt{\frac{9gh}{2}}\cos\theta$	$v_y =$
$a_x = 0$	$a_y = -g$
$s_x = 3h$	$s_y = -h$
$t_x = t$	$t_y = t.$

We first use $s_x = u_x t$ to get

$$
\begin{aligned}
3h &= \sqrt{\frac{9gh}{2}}\cos\theta t \\
\Rightarrow \frac{3h}{\sqrt{\frac{9gh}{2}}\cos\theta} &= t \\
\Rightarrow \sqrt{\frac{9h^2}{\frac{9gh}{2}}}\frac{1}{\cos\theta} &= t \\
\Rightarrow \sqrt{\frac{2h}{g}}\frac{1}{\cos\theta} &= t.
\end{aligned}
$$

Using this value for t in the equation $s = ut + \frac{1}{2}at^2$ in the y variables we get

$$\begin{aligned}
-h &= \sqrt{\frac{9gh}{2}}\sin\theta\sqrt{\frac{2h}{g}}\frac{1}{\cos\theta} - \frac{g}{2}\left(\sqrt{\frac{2h}{g}}\frac{1}{\cos\theta}\right)^2 \\
-h &= \sqrt{9h^2}\frac{\sin\theta}{\cos\theta} - \frac{g}{2}\frac{2h}{g}\frac{1}{\cos^2\theta} \\
\Rightarrow -h &= 3h\tan\theta - h\frac{1}{\cos^2\theta} \\
\Rightarrow -1 &= 3\tan\theta - \left(1+\tan^2\theta\right) \\
\Rightarrow \tan^2\theta - 3\tan\theta &= 0 \\
\Rightarrow \tan\theta\left(\tan\theta - 3\right) &= 0 \\
\Rightarrow \tan\theta &= 0, 3 \\
\Rightarrow \theta &= 0^\circ, 72^\circ.
\end{aligned}$$

(ii) If $\tan\theta = 0, 3$ then $\cos\theta = 1, \frac{1}{\sqrt{10}}$ so that

$$\begin{aligned}
t &= \sqrt{\frac{2h}{g}}\frac{1}{\cos\theta} \\
&= \sqrt{\frac{2h}{g}}, \sqrt{\frac{2h}{g}}\sqrt{10} \\
&= \sqrt{\frac{2h}{g}}, \sqrt{\frac{20h}{g}}.
\end{aligned}$$

Using (p, q)**:**

(i) Let $u = p\,\vec{i} + q\,\vec{j}$. We know the since the initial speed of the object is $\sqrt{\frac{9gh}{2}}$ that

$$\begin{aligned}
\sqrt{\frac{9gh}{2}} &= \sqrt{p^2+q^2} \\
\Rightarrow \frac{9gh}{2} &= p^2 + q^2.
\end{aligned}$$

This gives us one equation in p and q, admittedly in terms of h. To get the other equation we have the UVAST array

x-axis	y-axis
$u_x = p$	$u_y = q$
$v_x = p$	$v_y =$
$a_x = 0$	$a_y = -g$
$s_x = 3h$	$s_y = -h$
$t_x = t$	$t_y = t.$

We first use $s_x = u_x t$ to get

$$3h = pt$$
$$\Rightarrow \frac{3h}{p} = t$$

Using this value for t in the equation $s = ut + \frac{1}{2}at^2$ in the y variables we get

$$\begin{aligned}
-h &= q\left(\frac{3h}{p}\right) - \frac{g}{2}\left(\frac{3h}{p}\right)^2 \\
\Rightarrow -h &= \frac{3hq}{p} - \frac{g}{2}\frac{9h^2}{p^2} \\
\Rightarrow \frac{9gh^2}{2p^2} - h &= \frac{3hq}{p} \\
\Rightarrow \frac{9gh^2}{2p} - hp &= 3hq \\
\Rightarrow \frac{3gh}{2p} - \frac{p}{3} &= q.
\end{aligned}$$

Substituting this expression for q into the first equation we get

$$\begin{aligned}
\frac{9gh}{2} &= p^2 + \left(\frac{3gh}{2p} - \frac{p}{3}\right)^2 \\
&= p^2 + \frac{9g^2h^2}{2p^2} - gh + \frac{p^2}{9} \\
\Rightarrow 0 &= \frac{10}{9}p^2 - \frac{11gh}{2} + \frac{9g^2h^2}{2p^2} \\
\Rightarrow 0 &= \frac{10}{9}p^4 - \frac{11gh}{2}p^2 + \frac{9g^2h^2}{2}.
\end{aligned}$$

Recognising this as a quadratic in p^2 and using the quadratic formula we get

$$\begin{aligned}
p^2 &= \frac{\frac{11gh}{2} \pm \sqrt{\left(\frac{11gh}{2}\right)^2 - 4\frac{10}{9}\frac{9g^2h^2}{2}}}{2\frac{10}{9}} \\
&= \frac{\frac{11gh}{2} \pm \sqrt{\frac{121g^2h^2}{4} - 10g^2h^2}}{\frac{20}{9}} \\
&= \frac{\frac{11gh}{2} \pm \sqrt{\frac{81g^2h^2}{4}}}{\frac{20}{9}} \\
&= \frac{\frac{11gh}{2} \pm \frac{9gh}{2}}{\frac{20}{9}} \\
&= \frac{9gh}{2}, \frac{9gh}{20} \\
\Rightarrow p &= \sqrt{\frac{9gh}{2}}, \sqrt{\frac{9gh}{20}}.
\end{aligned}$$

Using either equation in p and q we can show that

$$q = 0,\ \sqrt{\frac{81gh}{20}}.$$

Therefore the angle of projection θ satisfies

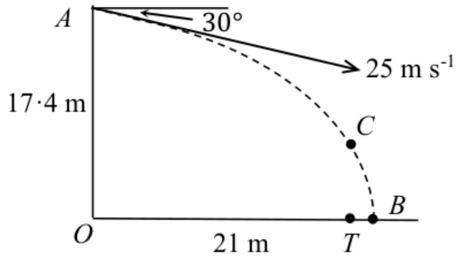
$$\begin{aligned}\tan\theta &= \frac{q}{p}\\ &= 0,\ \frac{\sqrt{\frac{81gh}{20}}}{\sqrt{\frac{9gh}{20}}}\\ &= 0,\ 3\\ \Rightarrow \theta &= 0, 72^\circ.\end{aligned}$$

(ii)

$$\begin{aligned}t &= \frac{3h}{p}\\ &= \frac{3h}{\sqrt{\frac{9gh}{2}}},\ \frac{3h}{\sqrt{\frac{9gh}{20}}}\\ &= \sqrt{\frac{9h^2}{\frac{9gh}{2}}},\ \sqrt{\frac{9h^2}{\frac{9gh}{20}}}\\ &= \sqrt{\frac{2h}{g}},\ \sqrt{\frac{20h}{g}}.\end{aligned}$$

Question — 2016 Q3 (a).

(a) A ball is thrown from a point A at a target T, which is on horizontal ground. The point A is 17·4 m vertically above the point O on the ground. The ball is thrown from A with speed 25 m s^{-1} at an angle of 30° below the horizontal. The distance OT is 21 m.
The ball misses the target and hits the ground at the point B, as shown in the diagram.
Find
(i) the time taken for the ball to travel from A to B
(ii) the distance TB.



The point C is on the path of the ball vertically above T.
(iii) Find the speed of the ball at C.

(i) We have the following UVAST array. Notice that this is a rare case of us knowing the initial velocity of the object. Also see that $u_y < 0$.

x-axis	y-axis
$u_x = 12.5\sqrt{3}$	$u_y = -12.5$
$v_x = 12.5\sqrt{3}$	$v_y =$
$a_x = 0$	$a_y = -g$
$s_x =$	$s_y = -17.4$
$t_x = t$	$t_y = t.$

Then

$$s_y = u_y t_y + \frac{1}{2} a_y t_y^2$$
$$\Rightarrow -17.4 = -12.5t - 4.9t^2$$
$$\Rightarrow 4.9t^2 + 12.5t - 17.4 = 0$$
$$\Rightarrow t = \frac{-12.5 \pm \sqrt{12.5^2 - 4(4.9)(-17.4)}}{2(4.9)}$$
$$= \cancel{-3.55}, 1 \text{ seconds.}$$

(ii) As

$$s_x = u_x t_x$$
$$= 12.5\sqrt{3}(1)$$
$$= 12.5\sqrt{3}$$

we have

$$|TB| = 12.5\sqrt{3} - 21$$
$$= 0.65 \text{ m.}$$

(iii) If we think of the journey ending at C we have the following UVAST array.

<u>x-axis</u>	<u>y-axis</u>
$u_x = 12.5\sqrt{3}$	$u_y = -12.5$
$v_x = 12.5\sqrt{3}$	$v_y =$
$a_x = 0$	$a_y = -g$
$s_x = 21$	$s_y =$
$t_x = t$	$t_y = t.$

Then

$$\begin{aligned} s_x &= u_x t_x \\ \Rightarrow 21 &= 12.5\sqrt{3}t \\ \Rightarrow 0.97 &= t. \end{aligned}$$

Then

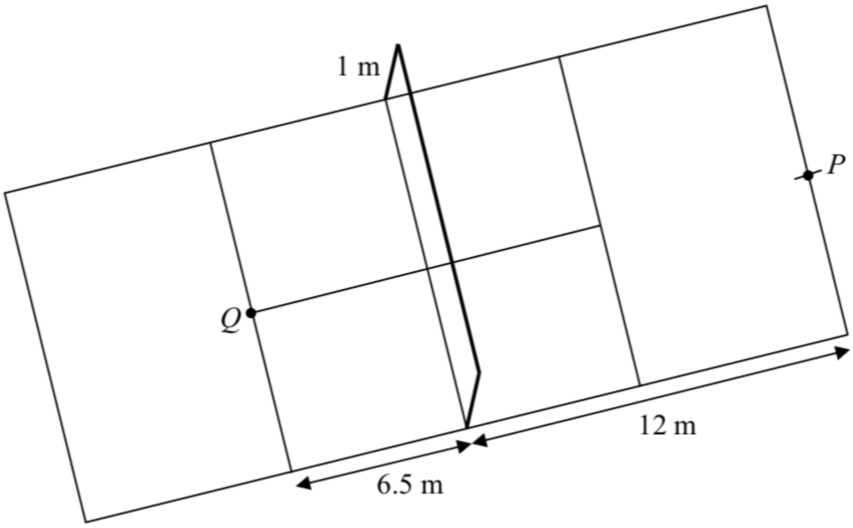
$$\begin{aligned} v_y &= u_y + a_y t_y \\ &= -12.5 - 9.8(0.97) \\ &= -22.006. \end{aligned}$$

See how we were asked for the **speed**, not the velocity at C.

$$\begin{aligned} |v| &= \sqrt{v_x^2 + v_y^2} \\ &= \sqrt{(12.5\sqrt{3})^2 + 22.006^2} \\ &= 30.87 \text{ m/s}. \end{aligned}$$

Question — 2015 Q3 (a).

(a) A tennis player, standing at P, serves a tennis ball from a height of 3 m to strike the court at Q. The speed of serve is 50 m s^{-1} at an angle β to the horizontal.



(i) Find the two possible values of $\tan\beta$.

(ii) For each value of $\tan\beta$ find the time, t, it takes the ball to reach Q.

(iii) If the tennis player chooses the smaller value of t, by what distance does the ball clear the net?

Using (u,θ):

(i) This is a standard target practice problem, with the following UVAST array. Notice that $s_y = -3$, not 0.

<u>x-axis</u>	<u>y-axis</u>
$u_x = 50\cos\beta$	$u_y = 50\sin\beta$
$v_x = 50\cos\beta$	$v_y =$
$a_x = 0$	$a_y = -g$
$s_x = 18.5$	$s_y = -3$
$t_x = t$	$t_y = t.$

We first use $s_x = u_x t$ to get

$$18.5 = 50\cos\beta t$$
$$\Rightarrow \frac{18.5}{50\cos\theta} = t.$$

Using this value for t in the equation $s = ut + \frac{1}{2}at^2$ in the y variables we get

$$
\begin{aligned}
-3 &= 50\sin\beta\left(\frac{18.5}{50\cos\theta}\right) - \frac{g}{2}\left(\frac{18.5}{50\cos\theta}\right)^2 \\
\Rightarrow -3 &= \frac{18.5\sin\beta}{\cos\beta} - 4.9\left(\frac{0.1369}{\cos^2\beta}\right) \\
\Rightarrow -3 &= 18.5\tan\beta - 0.67081\left(1+\tan^2\beta\right) \\
\Rightarrow 0.67081\tan^2\beta - 18.5\tan\beta - 2.32919 &= 0 \\
\Rightarrow \tan\beta &= \frac{18.5 \pm \sqrt{18.5^2 - 4(0.67081)(-2.32919)}}{2(0.67081)} \\
&= -0.13, 27.7.
\end{aligned}
$$

(ii) From part (i),

$$
\begin{aligned}
&\tan\beta = -0.13, 27.7 \\
&\Rightarrow \beta = -7^\circ, 88^\circ.
\end{aligned}
$$

Then

$$
\begin{aligned}
t &= \frac{18.5}{50\cos\beta} \\
&= \frac{18.5}{50\cos(-7^\circ)}, \frac{18.5}{50\cos 88^\circ} \\
&= 0.37, 10.6 \text{ seconds.}
\end{aligned}
$$

(iii) We have the following UVAST array, with the initial velocity calculated using $\beta = -7^\circ$.

x-axis	y-axis
$u_x = 49.63$	$u_y = 6.09$
$v_x = 49.63$	$v_y =$
$a_x = 0$	$a_y = -g$
$s_x = 12$	$s_y =$
$t_x = t$	$t_y = t.$

$$
\begin{aligned}
s_x &= u_x t_x \\
\Rightarrow 12 &= 49.63t \\
\Rightarrow 0.24 &= t.
\end{aligned}
$$

Then

$$
\begin{aligned}
s_y &= u_y t_y + \frac{1}{2}a_y t_y^2 \\
&= 6.09(0.24) - 4.9(0.24)^2 \\
&= 1.18 \text{ m}
\end{aligned}
$$

so that it clears the net by

$$1.18 - 1 = 0.18 \text{ m}.$$

Using (p,q):

(i) Let $u = p\vec{i} + q\vec{j}$. We know the since the initial speed of the object is 50 that

$$\begin{aligned} 50 &= \sqrt{p^2+q^2} \\ \Rightarrow 2500 &= p^2+q^2. \end{aligned}$$

This gives us one equation in p and q. To get the other equation we have the UVAST array

$$\begin{array}{ll} \underline{x\text{-axis}} & \underline{y\text{-axis}} \\ u_x = p & u_y = q \\ v_x = p & v_y = \\ a_x = 0 & a_y = -g \\ s_x = 18.5 & s_y = -3 \\ t_x = t & t_y = t. \end{array}$$

We first use $s_x = u_x t$ to get

$$\begin{aligned} 18.5 &= pt \\ \Rightarrow \frac{18.5}{p} &= t \end{aligned}$$

Using this value for t in the equation $s = ut + \frac{1}{2}at^2$ in the y variables we get

$$\begin{aligned} -3 &= q\left(\frac{18.5}{p}\right) - \frac{g}{2}\left(\frac{18.5}{p}\right)^2 \\ \Rightarrow -3 &= \frac{18.5q}{p} - 4.9\frac{342.25}{p^2} \\ \Rightarrow \frac{1677.025}{p^2} - 3 &= \frac{18.5q}{p} \\ \Rightarrow \frac{90.65}{p} - \frac{6p}{37} &= q. \end{aligned}$$

Substituting this expression for q into the first equation we get

$$\begin{aligned} 2500 &= p^2 + \left(\frac{90.65}{p} - \frac{6p}{37}\right)^2 \\ &= p^2 + \frac{8217.4225}{p^2} - 29.4 + \frac{36p^2}{1369} \\ \Rightarrow 0 &= \frac{1405}{1369}p^2 - 2529.4 + \frac{8217.4225}{p^2} \\ \Rightarrow 0 &= \frac{1405}{1369}p^4 - 2529.4p^2 + 8217.4225. \end{aligned}$$

Recognising this as a quadratic in p^2 and using the quadratic formula we get

$$\begin{aligned} p^2 &= 3.25, 2461.34 \\ \Rightarrow p &= 1.8, 49.61. \end{aligned}$$

Then

$$\begin{aligned} q &= \frac{90.65}{p} - \frac{6p}{37} \\ &= \frac{90.65}{1.8} - \frac{6(1.8)}{37}, \frac{90.65}{49.61} - \frac{6(49.61)}{37} \\ &= 50.07, -6.22. \end{aligned}$$

Ignoring the fact that $q < 50$ and chalking up to rounding error, we get

$$\begin{aligned} \tan\beta &= \frac{q}{p} \\ &= \frac{50.07}{1.8}, \frac{-6.22}{49.61} \\ &= 27.82, -0.13. \end{aligned}$$

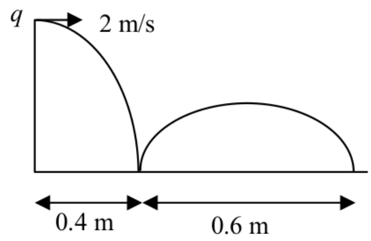
(ii) Same as with u, θ.

(iii) Same as with u, θ.

Question — 2005 Q3 (a).

(a) A ball is projected horizontally from a point q above a smooth horizontal plane with speed 2 m/s.
The ball first hits the plane at a point whose horizontal displacement from q is 0.4 m.
The ball next strikes the plane at a horizontal displacement of 1 m from q.
The coefficient of restitution between the ball and the plane is e.

Find the value of e.



We have the following UVAST array for both parts.

First Part		Second Part		Extra Equations
x-axis	y-axis	x-axis	y-axis	
$u_{x1} = 2$	$u_{y1} = 0$	$u_{x2} = 2$	$u_{y2} =$	
$v_{x1} = 2$	$v_{y1} =$	$v_{x2} = 2$	$v_{y2} =$	$u_{y2} = -ev_{y1}$
$a_{x1} = 0$	$a_{y1} = -g$	$a_{x2} = 0$	$a_{y2} = -g$	
$s_{x1} = 0.4$	$s_{y1} =$	$s_{x2} = 0.6$	$s_{y2} = 0$	
$t_{x1} = t_1$	$t_{y1} = t_1$	$t_{x2} = t_2$	$t_{y2} = t_2$	

In the first part we have

$$\begin{aligned} s_{x1} &= u_{x1}t_{x1} \\ \Rightarrow 0.4 &= 2t_1 \\ \Rightarrow 0.2 &= t_1 \end{aligned}$$

so that

$$\begin{aligned} v_{y1} &= u_{y1} + a_{y1}t_{y1} \\ &= -1.96. \end{aligned}$$

Then

$$\begin{aligned} u_{y2} &= -e(-1.96) \\ &= 1.96e. \end{aligned}$$

In the second part,

$$\begin{aligned} s_{x2} &= u_{x2}t_{x2} \\ \Rightarrow 0.6 &= 2t_2 \\ \Rightarrow 0.3 &= t_2. \end{aligned}$$

Then

$$\begin{aligned} s_{y2} &= u_{y2}t_{y2} + \frac{1}{2}a_{y2}t_{y2}^2 \\ \Rightarrow 0 &= 1.96e(0.3) - 4.9(0.3)^2 \\ \Rightarrow 0.441 &= 0.588e \\ \Rightarrow 0.75 &= e. \end{aligned}$$