Linear Motion Exam Question Solutions

Note These exam questions are answered in reverse chronological order as they appear in exam papers; 2023 paper, Sample paper, 2022 (deferred), 2022, and so on back to 2015. They are answered in the style described in my notes. All questions from the old syllabus papers are included as they are relevant to the new syllabus, with the possible exception of 2015 Q1 for which explanation for its inclusion is given.

Example — 2023 Q5 (b).

(b) Áine travels by car from her house to work each morning. On Monday morning she starts her car and accelerates uniformly for 40 s to a speed of 22.5 m s^{-1}. Áine then travels at this speed for 8 minutes until decelerating uniformly to rest at her work. She reaches her work at exactly 08:30.

On Tuesday morning Áine leaves her house 140 s later than the day before. She takes the same route to work. She starts her car and accelerates at 1.5 m s^{-2} for 20 s, then maintains this steady speed for 6 minutes before decelerating uniformly to rest at her work. She again reaches her work at exactly 08:30.

Calculate the time when Áine leaves her house on Tuesday morning.

Letting the Monday journey be Journey A and the Tuesday journey be Journey B,

	Journey A	
First Part	Second Part	Third Part
$u_{A1} = 0$	$u_{A2} = 22.5$	$u_{A3} = 22.5$
$v_{A1} = 22.5$	$v_{A2} = 22.5$	$v_{A3} = 0$
$a_{A1} =$	$a_{A2} = 0$	$a_{A3} =$
$s_{A1} =$	$s_{A2} =$	$s_{A3} =$
$t_{A1} = 40$	$t_{A2} = 480$	$t_{A3} =$

Journey B

First Part	Second Part	Third Part
$u_{B1} = 0$	$u_{B2} =$	$u_{B3} =$
$v_{B1} =$	$v_{B2} =$	$v_{B3} = 0$
$a_{B1} = 1.5$	$a_{B2} = 0$	$a_{B3} =$
$s_{B1} =$	$s_{B2} =$	$s_{B3} =$
$t_{B1} = 20$	$t_{B2} = 360$	$t_{B3} =$

Extra Equations

$$40 + 480 + t_{A3} = 20 + 360 + t_{B3} + 140 \qquad \text{(leaves later on Tuesday)}$$

$$s_{A1} + s_{A2} + s_{A3} = s_{B1} + s_{B2} + s_{B3} \qquad \text{(same distance)}$$

$$v_{B1} = u_{B2} = v_{B2} = u_{B3}$$

Some values can be calculated immediately (in brackets below), simplifying things.

Journey A

First Part	Second Part	Third Part
$u_{A1} = 0$	$u_{A2} = 22.5$	$u_{A3} = 22.5$
$v_{A1} = 22.5$	$v_{A2} = 22.5$	$v_{A3} = 0$
$a_{A1} =$	$a_{A2} = 0$	$a_{A3} =$
$s_{A1} = (450)$	$s_{A2} = (10800)$	$s_{A3} =$
$t_{A1} = 40$	$t_{A2} = 480$	$t_{A3} =$

Journey B

First Part	Second Part	Third Part
$u_{B1} = 0$	$u_{B2} = (30)$	$u_{B3} = (30)$
$v_{B1} = (30)$	$v_{B2} = (30)$	$v_{B3} = 0$
$a_{B1} = 1.5$	$a_{B2} = 0$	$a_{B3} =$
$s_{B1} = (300)$	$s_{B2} = (10800)$	$s_{B3} =$
$t_{B1} = 20$	$t_{B2} = 360$	$t_{B3} =$

Extra Equations

$$40 + 480 + t_{A3} = 20 + 360 + t_{B3} + 140$$

$$(450) + (10800) + s_{A3} = (300) + (10800) + s_{B3}$$

$$v_{B1} = u_{B2} = v_{B2} = u_{B3}$$

Letting $t_{B3} = t$, our first extra equation gives

$$40 + 480 + t_{A3} = 20 + 360 + t_{B3} + 140$$
$$\Rightarrow t_{A3} = t.$$

Letting $s_{B3} = s$, our second extra equation gives

$$s_{A1} + s_{A2} + s_{A3} = s_{B1} + s_{B2} + s_{B3}$$
$$\Rightarrow 450 + 10800 + s_{A3} = 300 + 10800 + s$$
$$\Rightarrow s_{A3} = s - 150.$$

Then applying $s = \left(\frac{u+v}{2}\right)t$ to the third column of both journeys,

$$\begin{aligned} s_{A3} &= \left(\frac{u_{A3}+v_{A3}}{2}\right)t_{A3} \\ \Rightarrow s-150 &= 11.25t, \end{aligned}$$

and

$$\begin{aligned} s_{B3} &= \left(\frac{u_{B3}+v_{B3}}{2}\right)t_{B3} \\ \Rightarrow s &= 15t. \end{aligned}$$

Solving the system of simultaneous equations

$$\begin{aligned} s-150 &= 11.25t \\ s &= 15t \end{aligned}$$

gives

$$t = 40,$$

so that

$$\begin{aligned} t_{B1}+t_{B2}+t_{B3} &= 20+360+40 \\ &= 420 \text{ seconds} \\ &= 7 \text{ minutes} \end{aligned}$$

so that she leaves her house at 8:23.

Example — Sample Q3 (b).

(b) Two athletes, Brian and Clara, are taking part in a relay race. Brian is preparing to hand over the baton to Clara. During the hand-over of the baton the athletes need to be running in the same straight line and at the same velocity.

As Brian approaches Clara's position at a constant speed of 11 m s^{-1}, Clara starts running from rest with constant acceleration f.

A short time later Brian begins to decelerate at 2 m s^{-2}.

Clara receives the baton 2.5 s after she starts running.

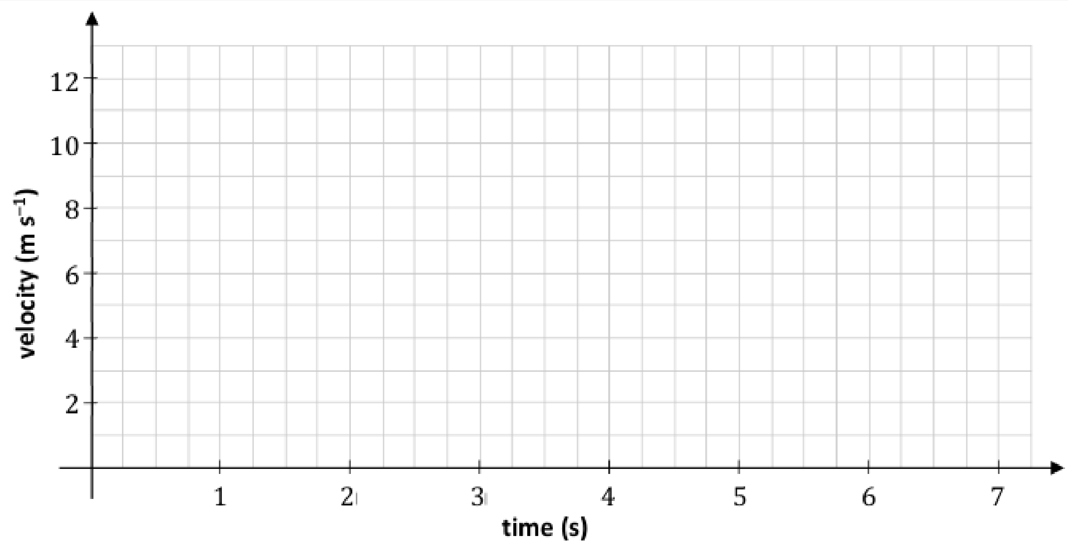
The baton is exchanged when Clara is 75 cm ahead of Brian and when both athletes have a speed of 8 m s^{-1}.

After the baton is exchanged, Brian continues to decelerate at 2 m s^{-2} until he comes to rest. Clara continues to accelerate at f until she reaches her maximum speed of 12 m s^{-1}, which she then maintains.

(i) Calculate the time it takes for Brian to decelerate before he exchanges the baton.

(ii) Using the axes below, draw an *accurate* velocity-time graph for the motion of each runner. Time is measured from the instant that Clara begins to run.

Relevant calculations should be shown in the space below.



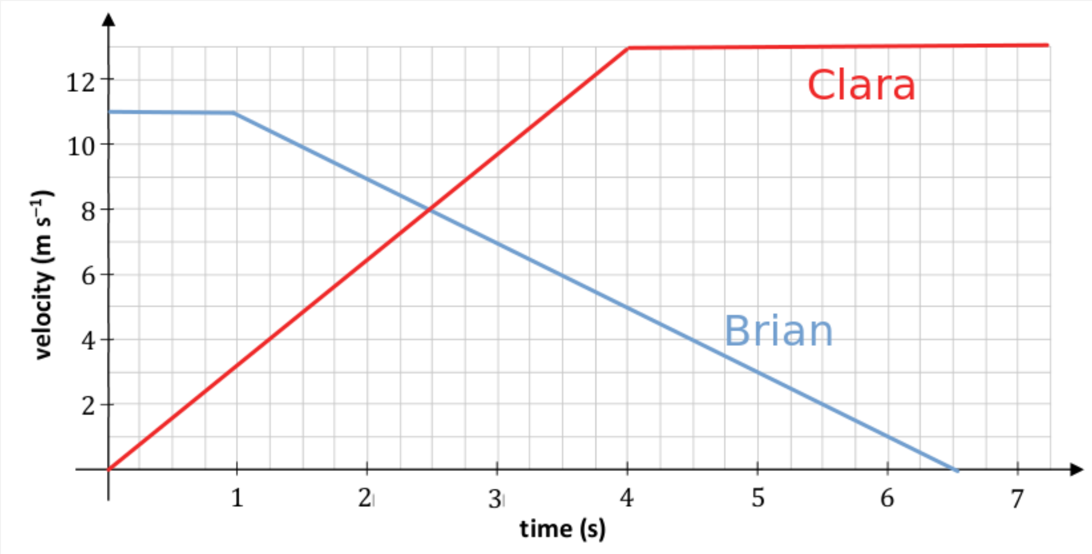
(iii) Calculate the distance between the two athletes when Clara begins to run.

(i) Letting the journeys for both athletes start when Clara starts running and end when the baton is passed,

Brian Part 1	Brian Part 2	Clara	Extra Equations
$u_{B1} = 11$	$u_{B2} = 11$	$u_C = 0$	$t_{B1} + t_{B2} = 2.5$
$v_{B1} = 11$	$v_{B2} = 8$	$v_C = 8$	
$a_{B1} = 0$	$a_{B2} = -2$	$a_C = f$	
$s_{B1} =$	$s_{B2} =$	$s_C =$	
$t_{B1} =$	$t_{B2} =$	$t_C = 2.5$	

We can quickly calculate that $t_{B2} = 1.5$ seconds from the second column.

(ii) We can also calculate that $t_{B1} = 1$ from the extra equation, giving the following graph. We don't explicitly need the times that Clara stops accelerating or that Brian reaches rest (although you can calculate them as 3.75 seconds and 5.5 seconds if you want) as we know to draw a straight line through $(2.5, 8)$.



(iii) Using the UVAST array from part (i), we can quickly calculate that

$$s_{B1} = 11$$
$$s_{B2} = 14.25$$
$$s_C = 10.$$

So Brian travelled $11 + 14.25 = 25.25$ m in the 2.5 seconds, so that Clara is $25.25 + 0.75 = 26$ m ahead of his starting position when the baton is passed. Therefore when she starts running she is $26 - 10 = 16$ m in front of Brian's starting position.

Example — 2022 (Deferred) Q1.

(a) Two cars, A and B, travel along a straight level road in opposite directions. A passes point P with speed 4 m s^{-1} and uniform acceleration 2 m s^{-2}. Three seconds later B passes point Q with speed 5 m s^{-1} and uniform acceleration 4 m s^{-2}.

The distance from P to Q is 1143 m.

The cars meet t seconds after A passes P.

(i) Find the value of t.

(ii) Find the distance from P to the meeting point.

(iii) Find the distance between the cars when A is 160 m from the meeting point, before the cars meet.

(b) An object falls vertically, from rest, from a height h metres. It travels $\frac{15}{64}h$ metres during its final second of motion before hitting the ground.

(i) Find the time it takes to fall to the ground.

(ii) Find the value of h.

(a) (i)

A	B	Extra Equations
$u_A = 4$	$u_B = 5$	~~$t_B = t_A - 3$~~
$v_A =$	$v_B =$	~~$s_A + s_B = 1143$~~
$a_A = 2$	$a_B = 4$	
$s_A = s$	$s_B = 1143 - s$	
$t_A = t$	$t_B = t - 3$	

From the first column,

$$s_A = u_A t_A + \frac{1}{2} a_A t_A^2$$
$$\Rightarrow s = 4t + t^2.$$

From the seconds column,

$$s_B = u_B t_B + \frac{1}{2} a_B t_B^2$$
$$\Rightarrow 1143 - s = 5(t-3) + 2(t-3)^2$$
$$\Rightarrow 1143 - (4t + t^2) = 5t - 15 + 2(t^2 - 6t + 9)$$
$$\Rightarrow 1143 - 4t - t^2 = 5t - 15 + 2t^2 - 12t + 18$$
$$\Rightarrow 0 = 3t^2 - 3t - 1140$$
$$\Rightarrow 0 = t^2 - t - 380$$
$$\Rightarrow t = 20, \cancel{-19} \text{ seconds.}$$

(ii) almost immediately from part (i),

$$\begin{aligned} s &= 4t + t^2 \\ &= 4(20) + 20^2 \\ &= 480 \text{ m.} \end{aligned}$$

(iii) Setting up a new UVAST array,

<u>A</u>	<u>B</u>
$u_A = 4$	$u_B = 5$
$v_A =$	$v_B =$
$a_A = 2$	$a_B = 4$
$s_A = 320$	$s_B =$
$t_A = t$	$t_B = t - 3$

where t is different to the t from part (i).

$$\begin{aligned} s_A &= u_A t_A + \frac{1}{2} a_A t_A^2 \\ \Rightarrow 320 &= 4t + t^2 \\ \Rightarrow 0 &= t^2 + 4t - 320 \\ \Rightarrow t &= 16, \cancel{-20}. \end{aligned}$$

Then

$$\begin{aligned} s_B &= u_B t_B + \frac{1}{2} a_B t_B^2 \\ &= 5(13) + 2(13)^2 \\ &= 403. \end{aligned}$$

The distance between the cars is then

$$1143 - 320 - 403 = 420 \text{ m.}$$

(b) (i)

<u>First Part</u>	<u>Second Part</u>	<u>Extra Equations</u>
$u_1 = 0$	$u_2 = v$	~~$v_1 = u_2$~~
$v_1 = v$	$v_2 =$	~~$s_1 + s_2 = h$~~
$a_1 = g$	$a_2 = g$	
$s_1 = \frac{49}{64} h$	$s_2 = \frac{15}{64} h$	
$t_1 =$	$t_2 = 1$	

$$\begin{aligned} v_1^2 &= u_1^2 + 2a_1 s_1 \\ \Rightarrow v^2 &= 0^2 + 2g\left(\frac{49}{64}h\right) \\ \Rightarrow v^2 &= \frac{49}{32}gh \\ \Rightarrow \frac{32v^2}{49g} &= h. \end{aligned}$$

$$\begin{aligned} s_2 &= u_2 t_2 + \frac{1}{2}a_2 t_2^2 \\ \Rightarrow \frac{15}{64}\frac{32v^2}{49g} &= v + \frac{1}{2}g \\ \Rightarrow \frac{15}{98g}v^2 - v - \frac{1}{2}g &= 0 \\ \Rightarrow v &= \frac{1 \pm \sqrt{1^2 - 4\left(\frac{15}{98g}\right)\left(-\frac{1}{2}g\right)}}{2\left(\frac{15}{98g}\right)} \\ &= \frac{1 \pm \sqrt{1 + \frac{15}{49}}}{\frac{15}{98g}} \\ &= \frac{1 \pm \sqrt{\frac{64}{49}}}{\frac{15}{49g}} \\ &= \frac{1 \pm \frac{8}{7}}{\frac{15}{49g}} \\ &= \frac{\frac{15}{7}}{\frac{15}{49g}} \quad (\text{ as } t > 0) \\ &= 7g. \end{aligned}$$

Therefore

$$\begin{aligned} v_1 &= u_1 + a_1 t_1 \\ \Rightarrow 7g &= 0 + gt \\ \Rightarrow 7 &= t. \end{aligned}$$

Therefore the total time until it hits the ground is $7 + 1 = 8$ seconds.

(ii) From part (i), if $v = 7g$ then

$$\begin{aligned} h &= \frac{32(7g)^2}{49g} \\ &= 32g \text{ m.} \end{aligned}$$

Example — 2022 Q1.

(a) A train takes 40 minutes to travel from rest at station A to rest at station B. The distance between the stations is 20 km. The train left station A at 10:00. At 10:15 the speed of the train was 32 km h^{-1} and at 10:30 the speed was 48 km h^{-1}.

The speed of 48 km h^{-1} was maintained until the brakes were applied, causing a uniform deceleration which brought the train to rest at B.

During the first and second 15-minute intervals the accelerations were constant.

(i) Draw a speed-time graph of the motion.

(ii) Find the time taken for the first 16 km.

(iii) Find the deceleration of the train.

(b) A ball E is thrown vertically upwards with a speed of 42 m s^{-1}.

T (< 8) seconds later another ball, F, is thrown vertically upwards from the same point with the same initial speed.

(i) Find where ball E is after 5 s and the total distance it has travelled in this time.

(ii) Prove that when E and F collide, they will each be travelling with speed $\frac{1}{2}gT$.

(a) (i) Our speed-time graph looks as follows.

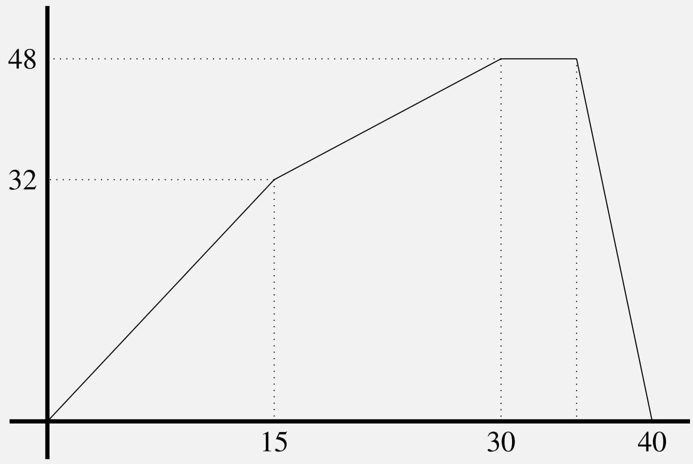


Figure 1

We don't know for how long the constant speed of 48 km/hr is maintained, so we don't have a time marked between 30 and 40. We also used minutes on the horizontal axis for the speed-time graph. This is fine as the alternative is awkward decimal numbers for the fractions of hours, but if we used this graph for any mathematics (such as finding the area under the curve) we would need to change the times to hours.

(ii) The question mentions minutes but speed is measured in km/hr. It's easier to convert time to different units than speed, so we will use km and hours in this answer.

Our initial UVAST array is as follows.

First Part	Second Part	Third Part	Fourth Part	Extra Equations
$u_1 = 0$	$u_2 = 32$	$u_3 = 48$	$u_4 = 48$	$t_3 + t_4 = \frac{1}{6}$
$v_1 = 32$	$v_2 = 48$	$v_3 = 48$	$v_4 = 0$	$s_1 + s_2 + s_3 + s_4 = 20$
$a_1 =$	$a_2 =$	$a_3 = 0$	$a_4 =$	
$s_1 =$	$s_2 =$	$s_3 =$	$s_4 =$	
$t_1 = \frac{1}{4}$	$t_2 = \frac{1}{4}$	$t_3 =$	$t_4 =$	

One of our extra equations has four variables, but we can find some of the variables immediately before we replace this extra equation with variables.

$$\begin{aligned} s_1 &= \left(\frac{u_1 + v_1}{2}\right) t_1 \\ \Rightarrow s_1 &= \left(\frac{0 + 32}{2}\right) \frac{1}{4} \\ &= 4. \end{aligned}$$

We can similarly find that $s_2 = 10$. Then

$$s_1 + s_2 + s_3 + s_4 = 20$$

becomes

$$\begin{aligned} 4 + 10 + s_3 + s_4 &= 20 \\ \Rightarrow s_3 + s_4 &= 6. \end{aligned}$$

Letting

$$\begin{aligned} t_3 &= t, \\ s_3 &= s, \end{aligned}$$

our UVAST array can then be written as

First Part	Second Part	Third Part	Fourth Part	Extra Equations
$u_1 = 0$	$u_2 = 32$	$u_3 = 48$	$u_4 = 48$	~~$t_3 + t_4 = \frac{1}{6}$~~
$v_1 = 32$	$v_2 = 48$	$v_3 = 48$	$v_4 = 0$	~~$s_1 + s_2 + s_3 + s_4 = 20$~~
$a_1 =$	$a_2 =$	$a_3 = 0$	$a_4 =$	
$s_1 = 4$	$s_2 = 10$	$s_3 = s$	$s_4 = 6 - s$	
$t_1 = \frac{1}{4}$	$t_2 = \frac{1}{4}$	$t_3 = t$	$t_4 = \frac{1}{6} - t$	

Solving for s and t in the usual way using simultaneous equations,

$$\begin{aligned} s_3 &= \left(\frac{u_3+v_3}{2}\right)t_3 \\ \Rightarrow s &= 48t. \\ s_4 &= \left(\frac{u_4+v_4}{2}\right)t_4 \\ \Rightarrow 6-s &= \left(\frac{48+0}{2}\right)\left(\frac{1}{6}-t\right) \\ &= 4-24t \\ \Rightarrow 2+24t &= s. \end{aligned}$$

Letting the two expressions for s equal each other,

$$\begin{aligned} 48t &= 2+24t \\ \Rightarrow 24t &= 2 \\ \Rightarrow t &= \frac{1}{12} \text{ hours} \\ \Rightarrow s &= 4 \text{ km}. \end{aligned}$$

Our UVAST array is now

First Part	Second Part	Third Part	Fourth Part	Extra Equations
$u_1 = 0$	$u_2 = 32$	$u_3 = 48$	$u_4 = 48$	$\cancel{t_3 + t_4 = \frac{1}{6}}$
$v_1 = 32$	$v_2 = 48$	$v_3 = 48$	$v_4 = 0$	$\cancel{s_1 + s_2 + s_3 + s_4 = 20}$
$a_1 =$	$a_2 =$	$a_3 = 0$	$a_4 =$	
$s_1 = 4$	$s_2 = 10$	$s_3 = \cancel{s}\ 4$	$s_4 = \cancel{6-s}\ 2$	
$t_1 = \frac{1}{4}$	$t_2 = \frac{1}{4}$	$t_3 = \cancel{t}\ \frac{1}{12}$	$t_4 = \cancel{\frac{1}{6}-t}\ \frac{1}{12}$	

Therefore we reach the 16 km mark during the third part of the journey. Specifically halfway through it, as we are at 14 km when we start this part of the journey and at 18 km at the end. Therefore the time taken for the first 16 km is

$$\frac{1}{4}+\frac{1}{4}+\frac{1}{2}\left(\frac{1}{12}\right) = \frac{13}{24} \text{ hours.}$$

Note that we can write our final answers in minutes (so in this case 32.5 minutes), we just can't mix units when doing our calculations. The question asks for the time taken, not the amount of minutes/hours so any units are fine.

(iii) The deceleration in the fourth part of the journey can easily be found using any equation of motion.

$$\begin{aligned} v &= u + at \\ 0 &= 48 + \frac{a}{12} \\ \Rightarrow a &= -576 \text{ km/hr}^2. \end{aligned}$$

So the train decelerates at a rate of 576 km/hr^2 in the final part of the journey.

(b) (i) This question doesn't require ball F, so if we're looking at the first 5 seconds of ball E's journey our UVAST array just looks like this.

$$\begin{aligned} u &= 42 \\ v &= \\ a &= -g \\ s &= \\ t &= 5. \end{aligned}$$

To get the position of the ball,

$$\begin{aligned} s &= ut + \frac{1}{2}at^2 \\ \Rightarrow s &= 42(5) - \frac{g}{2}(5)^2 \\ \Rightarrow s &= 87.5 \text{ m.} \end{aligned}$$

So ball E is 87.5 metres above where it was thrown. To find the total distance it has travelled, we need to know if it has changed direction, and if so what its greatest height was. See that

$$\begin{aligned} v &= u + at \\ \Rightarrow v &= 42 - 5g \\ &= -7. \end{aligned}$$

As final velocity is negative the ball has changed direction by time $t = 5$. Therefore we need to find the greatest height, which we can do using a different UVAST array:

$$\begin{aligned} u &= 42 \\ v &= 0 \\ a &= -g \\ s &= \\ t &= \end{aligned}$$

Then

$$\begin{aligned} v^2 &= u^2 + 2as \\ \Rightarrow 0 &= 42^2 - 2gs \\ \Rightarrow s &= \frac{42^2}{2g} \\ &= 90 \text{ m.} \end{aligned}$$

So the ball travelled up 90 m, then downwards $90 - 87.5 = 2.5$ m so that it's total distance travelled is 92.5 m.

(ii) The UVAST array below is for the journeys of balls E and F that start when they are thrown and end when they collide.

Ball E	Ball F	Extra Equations
$u_E = 42$	$u_F = 42$	$t_E = t_F + T$
$v_E =$	$v_F =$	$s_E = s_F$
$a_E = -g$	$a_F = -g$	
$s_E =$	$s_F =$	
$t_E =$	$t_F =$	

Letting $t_F = t$, $s_F = s$, our UVAST array can then be written as

Ball E	Ball F	Extra Equations
$u_E = 42$	$u_F = 42$	~~$t_E = t_F + T$~~
$v_E =$	$v_F =$	~~$s_E = s_F$~~
$a_E = -g$	$a_F = -g$	
$s_E = s$	$s_F = s$	
$t_E = t + T$	$t_F = t$	

Because the balls were thrown at the same speed, when they collide Ball E is travelling downwards and Ball F is travelling upwards. Therefore we want to show that

$$v_E = -\frac{1}{2}gT,$$
$$v_F = \frac{1}{2}gT.$$

We can treat T as an equation variable and s, t as non-equation variables. From the first column,

$$s_E = u_E t_E + \frac{1}{2}a_E t_E^2$$
$$\Rightarrow s = 42(t+T) - \frac{g}{2}(t+T)^2$$

so that we have s in terms of other variables. From the second column,

$$s_F = u_F t_F + \frac{1}{2}a_F t_F^2$$

$$\Rightarrow 42(t+T) - \frac{g}{2}(t+T)^2 = 42t - \frac{g}{2}t^2$$

$$\Rightarrow 42t + 42T - \frac{g}{2}(t^2 + 2tT + T^2) = 42t - \frac{g}{2}t^2$$

$$\Rightarrow 42t + 42T - \frac{g}{2}t^2 - gtT - \frac{g}{2}T^2 = 42t - \frac{g}{2}t^2$$

$$\Rightarrow 42T - gtT - \frac{g}{2}T^2 = 0$$

$$\Rightarrow 42T - \frac{g}{2}T^2 = gtT$$

$$\Rightarrow \frac{42}{g} - \frac{T}{2} = t.$$

Now that we have t in terms of only the equation variable T we can do similarly for s.

$$\begin{aligned}
s &= 42(t+T) - \frac{g}{2}(t+T)^2 \\
&= 42\left(\frac{42}{g} - \frac{T}{2} + T\right) - \frac{g}{2}\left(\frac{42}{g} - \frac{T}{2} + T\right)^2 \\
&= 42\left(\frac{42}{g} - \frac{T}{2}\right) - \frac{g}{2}\left(\frac{42}{g} - \frac{T}{2}\right)^2 \\
&= \frac{1764}{g} - 21T - \frac{g}{2}\left(\frac{1764}{g^2} - \frac{42}{g}T + \frac{T^2}{4}\right) \\
&= \frac{1764}{g} - 21T - \frac{882}{g} + 21T - \frac{gT^2}{8} \\
&= \frac{882}{g} - \frac{gT^2}{8}.
\end{aligned}$$

Our UVAST array now looks as follows.

Ball E	Ball F	Extra Equations
$u_E = 42$	$u_F = 42$	$\cancel{t_E = t_F + T}$
$v_E =$	$v_F =$	$\cancel{s_E = s_F}$
$a_E = -g$	$a_F = -g$	
$s_E = \cancel{s}\ \frac{882}{g} - \frac{gT^2}{8}$	$s_F = \cancel{s}\ \frac{882}{g} - \frac{gT^2}{8}$	
$t_E = \cancel{t+T}\ \frac{42}{g} + \frac{T}{2}$	$t_F = \cancel{t}\ \frac{42}{g} - \frac{T}{2}$	

We can now find v_E, v_F by applying any equation of motion which contains v to each column.

$$\begin{aligned}
v_E &= u_E + a_E t_E \\
\Rightarrow v_E &= 42 - g\left(\frac{42}{g} + \frac{T}{2}\right) \\
&= -\frac{gT}{2}.
\end{aligned}$$

Therefore the speed of ball E is

$$\left|-\frac{gT}{2}\right| = \frac{gT}{2},$$

as required. Similarly

$$\begin{aligned} v_F &= u_F + a_F t_F \\ \Rightarrow v_F &= 42 - g\left(\frac{42}{g} + \frac{T}{2}\right) \\ &= \frac{gT}{2}. \end{aligned}$$

Example — 2021 Q1.

(a) A ball is thrown vertically downwards from the top of a building of height h m. The ball passes the top half of the building in 1.2 s and takes a further 0.8 s to reach the bottom of the building.
Find

(i) the value of h

(ii) the speed of the ball at the bottom of the building.

(b) Car C, moving with uniform acceleration f passes a point P with speed u (> 0).
Two seconds later car D, moving in the same direction with uniform acceleration $2f$ passes P with speed $\frac{6}{5}u$. C and D pass a point Q together. The speeds of C and D at Q are 6.5 m s^{-1} and 9 m s^{-1} respectively.

(i) Show that C travels from P to Q in $(\frac{3}{2f} + 5)$ seconds.

(ii) Find the value of f.

(a) (i) Although the acceleration of the ball is constant we will split up the journey into two parts; from the top of the building to the middle and from the middle to the bottom. Letting $v_1, u_2 = v$ and letting the downward direction be positive our UVAST array looks as follows.

First Part	Second Part
$u_1 =$	$u_2 = v$
$v_1 = v$	$v_2 =$
$a_1 = g$	$a_2 = g$
$s_1 = \frac{h}{2}$	$s_2 = \frac{h}{2}$
$t_1 = 1.2$	$t_2 = 0.8$

Notice that the object was **thrown**, not dropped, so $u_1 \neq 0$.

We have two variables, v and h, so we want to create two simultaneous equations in two variables, one coming from each column as there is a blank subscripted variable in each column. In the first column,

$$s_1 = v_1 t_1 - \frac{1}{2} a_2 t_1^2$$
$$\Rightarrow \frac{h}{2} = 1.2v - \frac{g}{2}(1.2)^2$$
$$\Rightarrow \frac{h}{2} = 1.2v - 0.72g$$
$$\Rightarrow h = 2.4v - 1.44g.$$

In the second column,

$$
\begin{aligned}
s_2 &= u_2t_2 + \frac{1}{2}a_2t_2^2 \\
\Rightarrow \frac{h}{2} &= 0.8v + \frac{1}{2}g(0.8)^2 \\
\Rightarrow \frac{h}{2} &= 0.8v + 0.32g \\
\Rightarrow h &= 1.6v + 0.64g.
\end{aligned}
$$

Letting these expressions for h be equal we have

$$
\begin{aligned}
2.4v - 1.44g &= 1.6v + 0.64g \\
\Rightarrow 0.8v &= 2.08g \\
\Rightarrow v &= 2.6g \\
\Rightarrow h &= 2.4(2.6g) - 1.44g \\
\Rightarrow h &= 4.8g \text{ m.}
\end{aligned}
$$

(ii) To get the speed of the ball at the bottom of the building,

$$
\begin{aligned}
v &= u + at \\
\Rightarrow v_2 &= 2.6g + 0.8g \\
&= 3.4g \text{ m/s.}
\end{aligned}
$$

(b) (i) We could set up our UVAST array as follows.

Car C	Car D	Extra Equations
$u_C = u$	$u_D = \frac{6}{5}u$	$s_C = s_D$
$v_C = 6.5$	$v_D = 9$	$t_C = t_D + 2$
$a_C = f$	$a_D = 2f$	
$s_C =$	$s_D =$	
$t_C =$	$t_D =$	

Letting $s_C = s$, $t_C = t$, our UVAST array can then be written like this.

Car C	Car D	Extra Equations
$u_C = u$	$u_D = \frac{6}{5}u$	~~$s_C = s_D$~~
$v_C = 6.5$	$v_D = 9$	~~$t_C = t_D + 2$~~
$a_C = f$	$a_D = 2f$	
$s_C = s$	$s_D = s$	
$t_C = t$	$t_D = t - 2$	

We are being asked to show that

$$
t = \frac{3}{2f} + 5
$$

(why is why we let $t_C = t$ and not t_D) and so will solve this problem with t and f being the equation variables and u and s being the non-equation variables which we want to remove from the UVAST array. From the first column,

$$\begin{aligned} v_1 &= u_1 + a_1 t_1 \\ \Rightarrow 6.5 &= u + ft \\ \Rightarrow 6.5 - ft &= u. \end{aligned}$$

Also from the first column,

$$\begin{aligned} s_1 &= \left(\frac{u_1 + v_1}{2}\right) t_1 \\ \Rightarrow s &= \left(\frac{6.5 - ft + 6.5}{2}\right) t \\ &= 6.5t - 0.5ft^2. \end{aligned}$$

Our updated UVAST array looks as follows.

Car C	Car D	Extra Equations
$u_C = \cancel{u}\ 6.5 - ft$	$u_D = \cancel{\tfrac{6}{5}u}\ 7.8 - 1.2ft$	$\cancel{s_C = s_D}$
$v_C = 6.5$	$v_D = 9$	$\cancel{t_C = t_D + 2}$
$a_C = f$	$a_D = 2f$	
$s_C = \cancel{s}\ 6.5t - 0.5ft^2$	$s_D = \cancel{s}\ 6.5t - 0.5ft^2$	
$t_C = t$	$t_D = t - 2$	

Using any more equations on the first column won't help, and we want to avoid t^2 if we are trying to solve for t. Therefore we will choose to apply $v = u + at$ to the second column to get

$$\begin{aligned} v_2 &= u_2 + a_2 t_2 \\ \Rightarrow 9 &= 7.8 - 1.2ft + 2f(t-2) \\ &= 7.8 - 1.2ft + 2ft - 4f \\ \Rightarrow 1.2 + 4f &= 0.8ft \\ \Rightarrow \frac{1.2}{0.8f} + \frac{4}{0.8} &= t \\ \Rightarrow \frac{3}{2f} + 5 &= t, \end{aligned}$$

as required. Note again that our intermediate goal was just to get t and f in the same equation. Our final goal was to rewrite it in the form $t = \ldots$. As if by magic, we then get the equation asked for in the question.

(ii) Note that we now have t in terms of f. That means that we have every indexed variable in terms of f, and can rewrite the second column of our UVAST array (the only column we can still use) as

$$\begin{aligned}
&\underline{\text{Car D}}\\
u_D &= 7.8 - 1.2f\left(\frac{3}{2f} + 5\right)\\
&= 7.8 - 1.8 - 6f\\
&= 6 - 6f\\
v_D &= 9\\
a_D &= 2f\\
s_D &= 6.5\left(\frac{3}{2f} + 5\right) - 0.5f\left(\frac{3}{2f} + 5\right)^2\\
&= \frac{9.75}{f} + 32.5 - 0.5f\left(\frac{2.25}{f^2} + \frac{15}{f} + 25\right)\\
&= \frac{9.75}{f} + 32.5 - \frac{1.125}{f} - 7.5 - 12.5f\\
&= \frac{8.625}{f} + 25 - 12.5f\\
t_D &= \frac{1.5}{f} + 3.
\end{aligned}$$

Applying a different equation of motion to this column should yield an equation only containing f, which we should be able to solve. Avoiding equations with squares, mostly out of laziness,

$$\begin{aligned}
s &= \left(\frac{u+v}{2}\right)t\\
\Rightarrow \frac{8.625}{f} + 25 - 12.5f &= \left(\frac{6 - 6f + 9}{2}\right)\left(\frac{1.5}{f} + 3\right)\\
&= (7.5 - 3f)\left(\frac{1.5}{f} + 3\right)\\
&= \frac{11.25}{f} + 22.5 - 4.5 - 9f\\
\Rightarrow -\frac{2.625}{f} + 7 - 3.5f &= 0\\
\Rightarrow -2.625 + 7f - 3.5f^2 &= 0\\
\Rightarrow f &= 0.5, 1.5
\end{aligned}$$

by applying the quadratic formula.

The question says find the value of f, not the values. So how do we eliminate one of these values? Well $f = \frac{3}{2}$ implies

$$\begin{aligned}
u_D &= 6 - 6\left(\frac{3}{2}\right)\\
&= -3,
\end{aligned}$$

but we were told that $u > 0$, which implies $u_D > 0$. Therefore our answer is $f = \frac{1}{2}$. This is another example of a principle mentioned in class; if you are asked for the **value** of a variable and get two values, then there must be a reason that one of them should be discarded.

Example — 2020 Q1.

1. **(a)** A car is travelling on a straight level road at a uniform speed of 26 m s^{-1} when the driver notices a tractor 91·2 m ahead.
The tractor is travelling at a uniform speed of 6 m s^{-1} in the same direction as the car.
The driver of the car hesitates for t seconds before applying the brake.
The maximum deceleration of the car is 5 m s^{-2}.

Find the maximum value of t which would avoid a collision between the car and the tractor.

(b) A 60 gram mass is projected vertically upwards with an initial speed of 15 m s^{-1} and half a second later a 40 gram mass is projected vertically upwards from the same point with an initial speed of 22·65 m s^{-1}.

(i) Calculate the height at which the masses will collide.

The masses coalesce on colliding.

(ii) Find the greatest height which the combined mass will reach.

(a) Assume that the car and tractor "just about collide", i.e. that they collide but at this time their speeds are the same so that they separate immediately again as the car continues to slow down. Then taking the journey from when the car notices the tractor to when they collide

Car Part 1	Car Part 2	Tractor	Extra Equations
$u_{C1} = 26$	$u_{C2} = 26$	$u_T = 6$	$t_{C1} + t_{C2} = t_T$
$v_{C1} = 26$	$v_{C2} = 6$	$v_T = 6$	$s_{C1} + s_{C2} = s_T + 91.2$
$a_{C1} = 0$	$a_{C2} = -5$	$a_T = 0$	
$s_{C1} =$	$s_{C2} =$	$s_T =$	
$t_{C1} = t$	$t_{C2} =$	$t_T =$	

We can quickly calculate that $s_{C2} = 64$, $t_{C2} = 4$. Letting $s_{C1} = s$ and replacing our extra equations with algebra we get

Car Part 1	Car Part 2	Tractor	Extra Equations
$u_{C1} = 26$	$u_{C2} = 26$	$u_T = 6$	~~$t_{C1} + t_{C2} = t_T$~~
$v_{C1} = 26$	$v_{C2} = 6$	$v_T = 6$	~~$s_{C1} + s_{C2} = s_T + 91.2$~~
$a_{C1} = 0$	$a_{C2} = -5$	$a_T = 0$	
$s_{C1} = s$	$s_{C2} = 64$	$s_T = s - 27.2$	
$t_{C1} = t$	$t_{C2} = 4$	$t_T = t + 4$	

Then applying $s = ut + \frac{1}{2}at^2$ to the first and third column we get

$$s = 26t$$

and

$$\begin{aligned} s - 27.2 &= 6(t+4) \\ \Rightarrow s &= 6t + 24 + 27.2 \\ &= 6t + 51.2. \end{aligned}$$

Letting the s expressions be equal,

$$\begin{aligned} 26t &= 6t + 51.2 \\ \Rightarrow t &= 2.56 \text{ seconds.} \end{aligned}$$

(b) (i) If the 60 gram mass is the first object and the 40 gram mass is the second we have the following UVAST array.

First Object	Second Object	Extra Equations
$u_1 = 15$	$u_2 = 22.65$	~~$s_1 = s_2$~~
$v_1 =$	$v_D =$	~~$t_1 = t_2 + 0.5$~~
$a_1 = -g$	$a_D = -g$	
$s_1 = s$	$s_2 = s$	
$t_1 = t$	$t_2 = t - 0.5$	

Applying $s = ut + \frac{1}{2}at^2$ to both columns gives us

$$\begin{aligned} s_1 &= u_1 t_1 + \frac{1}{2} a_1 t_1^2 \\ \Rightarrow s &= 15t - \frac{g}{2} t^2 \end{aligned}$$

and

$$\begin{aligned} s_2 &= u_2 t_2 + \frac{1}{2} a_2 t_2^2 \\ \Rightarrow s &= 22.65(t - 0.5) - \frac{g}{2}(t - 0.5)^2 \\ &= 22.65t - 11.325 - 4.9(t^2 - t + 0.25) \\ &= -4.9t^2 + 27.55t - 12.55. \end{aligned}$$

Letting the s expressions equal each other

$$\begin{aligned} 15t - \frac{g}{2} t^2 &= -4.9t^2 + 27.55t - 12.55 \\ \Rightarrow 12.55 &= 12.55t \\ \Rightarrow 1 &= t \\ \Rightarrow s &= 15(1) - \frac{g}{2}(1)^2 \\ &= 10.1 \text{ m.} \end{aligned}$$

(ii) From (b) (i), we can quickly calculate $v_1 = 5.2$ and $v_2 = 17.75$ (remember that $t_2 = 0.5$, not 1). Using the Principle of Conservation of Momentum where after collision the objects have common velocity v,

$$\begin{aligned} m_1 u_1 + m_2 u_2 &= m_1 v_1 + m_2 v_2 \\ \Rightarrow (0.06)(5.2) + (0.04)(17.75) &= 0.06v + 0.04v \\ \Rightarrow 1.022 &= 0.1v \\ \Rightarrow 10.22 \text{ m/s} &= v. \end{aligned}$$

So the joint object has a velocity of 10.22 m/s upwards when it is 10.1 m above the ground (from part (b) (i)). Calculating how much higher it gets after collision,

$$\begin{aligned} u &= 10.22 \\ v &= 0 \\ a &= -g \\ s &= \\ t &= \end{aligned}$$

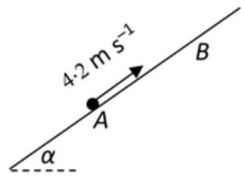
$$\begin{aligned} v^2 &= u^2 + 2as \\ \Rightarrow 0 &= 104.4484 - 19.6s \\ \Rightarrow s &= 5.329 \text{ m.} \end{aligned}$$

Therefore the greatest height that the joint mass reaches is

$$10.1 + 5.329 = 15.429 \text{ m.}$$

Example — 2019 Q1.

1. **(a)** A particle P, of mass 3 kg, is projected along a rough inclined plane from the point A with speed $4{\cdot}2$ m s^{-1}. The particle comes to instantaneous rest at B.
The plane is inclined at an angle α to the horizontal where $\tan\alpha = \frac{9}{40}$.
The coefficient of friction between the particle and the plane is $\frac{3}{20}$.



(i) Show that the deceleration of P is $\frac{15g}{41}$.

(ii) Find $|AB|$.

After reaching B the particle slides back down the plane.

(iii) Find the speed of P as it passes through A on its way back down the plane.

(b) Train A and Train B are on parallel tracks and travelling in opposite directions.
Train A starts from rest at Maynooth and accelerates uniformly at $0{\cdot}5$ m s^{-2} towards Leixlip to a speed of 25 m s^{-1}. It then continues at this constant speed.

At the same instant as train A is leaving Maynooth Train B passes through Leixlip heading towards Maynooth at a constant speed of 30 m s^{-1}.
Three minutes after leaving Leixlip train B starts to decelerate at $0{\cdot}25$ m s^{-2} and comes to rest at Maynooth.

(i) Find the distance between Maynooth and Leixlip.

(ii) At what distance from Maynooth do the trains meet?

After travelling at 25 m s^{-1} for a time, train A decelerates and comes to rest at Leixlip 36 seconds after train B stops at Maynooth.

(iii) Find the deceleration of train A.

(a) (i) If $\tan\alpha = \frac{9}{40}$, we can draw α inside the triangle

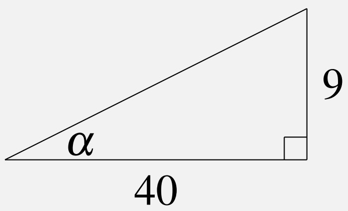


Figure 2

We can use Pythagoras' Theorem to calculate the hypotenuse to be 41, so that

$$\sin\alpha = \frac{9}{41}$$
$$\cos\alpha = \frac{40}{41}.$$

The forces acting on the particle are then as follows.

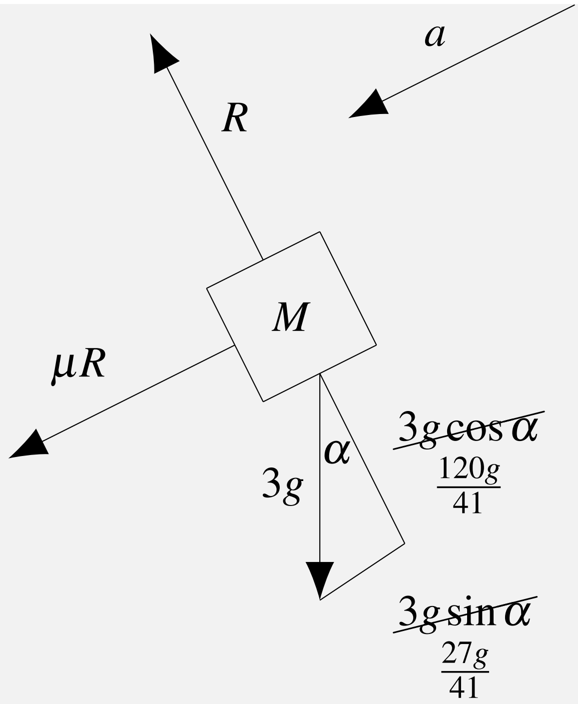


Figure 3

Note that although acceleration is down the slope, friction is also down the slope as **velocity** is up the slope. Then our two equations are

$$R = \frac{120g}{41},$$
$$\mu R + \frac{27g}{41} = 3a$$
$$\Rightarrow \left(\frac{3}{20}\right)\left(\frac{120g}{41}\right) + \frac{27g}{41} = 3a$$
$$\Rightarrow \frac{45g}{41} = 3a$$
$$\Rightarrow \frac{15g}{41} = a.$$

(ii) We have, with no trigonometry required, the UVAST array

$$u = 4.2$$
$$v = 0$$
$$a = -\frac{15g}{41}$$
$$s =$$
$$t =$$

Then

$$v^2 = u^2 + 2as$$
$$\Rightarrow 0 = 17.64 - \frac{30g}{41}s$$
$$\Rightarrow s = 2.46 \text{ m.}$$

Therefore $|AB| = 2.46$ m.

(iii) On the way down the frictional force reverses direction and so the forces acting on the particle are as follows.

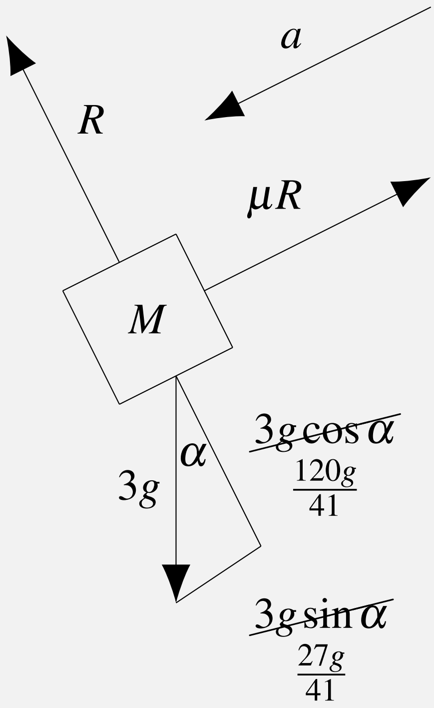


Figure 4

Therefore our equations are

$$R = \frac{120g}{41}$$
$$\frac{27g}{41} - \mu R = 3a$$
$$\Rightarrow \frac{27g}{41} - \left(\frac{3}{20}\right)\left(\frac{120g}{41}\right) = 3a$$
$$\Rightarrow \frac{9g}{41} = 3a$$
$$\Rightarrow \frac{3g}{41} = a.$$

Looking at the journey back from B to A we have the UVAST array

$$\begin{aligned} u &= 0 \\ v &= \\ a &= \frac{3g}{41} \\ s &= 2.46 \\ t &= \end{aligned}$$

Then

$$\begin{aligned} v^2 &= u^2 + 2as \\ &= 3.528 \\ \Rightarrow v &\approx 1.88 \text{ m/s.} \end{aligned}$$

(b) (i) Just looking at the journey from Leixlip to Maynooth for Train B,

First Part	Second Part
$u_1 = 30$	$u_2 = 30$
$v_1 = 30$	$v_2 = 0$
$a_1 = 0$	$a_2 = -0.25$
$s_1 =$	$s_2 =$
$t_1 = 180$	$t_2 =$

We can quickly calculate that $s_1 = 5,400$ and $s_2 = 1,800$ so that the distance from Maynooth to Leixlip is

$$s_1 + s_2 = 7200 \text{ m.}$$

(ii) Looking at the journey for both trains, starting when they leave Maynooth/Leixlip and ending when they meet we have the following UVAST array, which assumes (based on guesswork) that they meet after A stops accelerating but before B starts decelerating.

A Part 1	A Part 2	B	Extra Equations
$u_{A1} = 0$	$u_{A2} = 25$	$u_B = 30$	$t_{A1} + t_{A2} = t_B$
$v_{A1} = 25$	$v_{A2} = 25$	$v_B = 30$	$s_{A1} + s_{A2} + s_B = 7200$
$a_{A1} = 0.5$	$a_{A2} = 0$	$a_B = 0$	
$s_{A1} =$	$s_{A2} =$	$s_B =$	
$t_{A1} =$	$t_{A2} =$	$t_B =$	

We can quickly calculate $t_{A1} = 50$, $s_{A1} = 625$. Then letting $s_B = s$, $t_B = t$ our UVAST array becomes

A Part 1	A Part 2	B	Extra Equations
$u_{A1} = 0$	$u_{A2} = 25$	$u_B = 30$	~~$t_{A1} + t_{A2} = t_B$~~
$v_{A1} = 25$	$v_{A2} = 25$	$v_B = 30$	~~$s_{A1} + s_{A2} + s_B = 7200$~~
$a_{A1} = 0.5$	$a_{A2} = 0$	$a_B = 0$	
$s_{A1} = 625$	$s_{A2} = 6575 - s$	$s_B = s$	
$t_{A1} = 50$	$t_{A2} = t - 50$	$t_B = t$	

Then applying $s = ut + \frac{1}{2}at^2$ to the second and third column gives us

$$\begin{aligned} 6575 - s &= 25(t - 50) \\ \Rightarrow -s &= -6575 + 25t - 1250 \\ \Rightarrow s &= 7825 - 25t \end{aligned}$$

and

$$s = 30t.$$

Letting the s terms be equal,

$$\begin{aligned} 30t &= 7825 - 25t \\ \Rightarrow t &= 142.27 \\ \Rightarrow s &= 30(142.27) \\ &= 4268.18. \end{aligned}$$

$s = 4268.18$ is actually the distance from Leixlip to where they meet, so our answer is

$$7200 - 4268.18 = 2931.82 \text{ m}.$$

(iii) Using the UVAST array from (b) (i) we can quickly calculate $t_2 = 120$ so that it takes train B

$$t_1 + t_2 = 300$$

seconds to complete its trip. If it takes train A $300 + 36 = 336$ seconds to complete its trip we have the following UVAST array for its journey.

First Part	Second Part	Extra Equations	
$u_1 = 0$	$u_2 = 25$	$u_3 = 25$	$50 + t_2 + t_3 = 336$
$v_1 = 25$	$v_2 = 25$	$v_3 = 0$	$625 + s_2 + s_3 = 7200$
$a_1 = 0.5$	$a_2 = 0$	$a_3 =$	
$s_1 = 625$	$s_2 =$	$s_3 =$	
$t_1 = 50$	$t_2 =$	$t_3 =$	

Letting $t_3 = t$, $s_3 = s$, our UVAST array becomes

First Part	Second Part	Extra Equations	
$u_1 = 0$	$u_2 = 25$	$u_3 = 25$	~~$50 + t_2 + t_3 = 336$~~
$v_1 = 25$	$v_2 = 25$	$v_3 = 0$	~~$625 + s_2 + s_3 = 7200$~~
$a_1 = 0.5$	$a_2 = 0$	$a_3 =$	
$s_1 = 625$	$s_2 = 6575 - s$	$s_3 = s$	
$t_1 = 50$	$t_2 = 286 - t$	$t_3 = t$	

Applying the equation $s = \left(\frac{u+v}{2}\right) t$ to the second and third column gives us

$$\begin{aligned} 6575 - s &= 25(286 - t) \\ \Rightarrow -s &= -6575 + 7150 - 25t \\ \Rightarrow s &= 25t - 575. \end{aligned}$$

and

$$s = 12.5t.$$

Letting the s expressions be equal we get

$$\begin{aligned} 12.5t &= 25t - 575 \\ \Rightarrow t &= 46 \end{aligned}$$

so that

$$\begin{aligned} v_3 &= u_3 + a_3 t_3 \\ \Rightarrow 0 &= 25 + 46 a_3 \\ \Rightarrow a &\approx 0.54 \text{ m/s}^2. \end{aligned}$$

Example — 2018 Q1.

(a) A parcel rests on the horizontal floor of a van.
The van is travelling on a level road at 14 m s^{-1} .
It is brought to rest by a uniform application of the brakes.
The coefficient of friction between the parcel and the floor is $\frac{2}{5}$.
Show that the parcel is on the point of sliding forward on the floor of the van if the stopping distance is 25 m.

(b) A car C moves with uniform acceleration a from rest to a maximum speed u.
It then travels at uniform speed u.
Just as car C starts, it is overtaken by a car D moving in the same direction with constant speed $\frac{3u}{4}$.
Car C catches up with car D when car C has travelled a distance d.

(i) Show that, at the instant car C catches up with car D, car C has been travelling with speed u for a time $\frac{4d}{3u} - \frac{u}{a}$.

(ii) Find d in terms of u and a.

(a) We first find the deceleration of the van.

$$\begin{aligned} u &= 14 \\ v &= 0 \\ a &= \\ s &= 25 \\ t &= \end{aligned}$$

$$\begin{aligned} v^2 &= u^2 + 2as \\ \Rightarrow 0 &= 14^2 + 50a \\ \Rightarrow -50a &= 196 \\ \Rightarrow a &= -3.92 \text{ m/s}^2. \end{aligned}$$

Friction is acting to the back of the van, pulling the parcel back. Assume that the particle is not moving along the floor. Then it is also decelerating at 3.92 m/s^2 with the van. As friction (F), reaction force (R) and gravity are the only forces acting on the parcel, we have the following diagram.

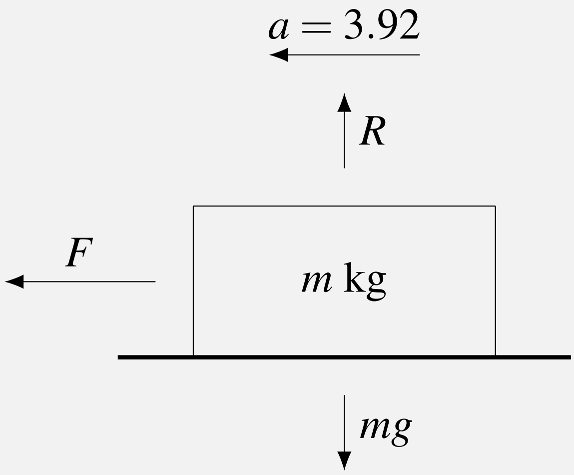


Figure 5

Then $R = mg$, and

$$F = ma$$
$$\Rightarrow F = 3.92m.$$

Also,

$$\mu R = \frac{2}{5}mg$$
$$= 3.92m,$$

so if the particle is not moving on the floor friction (F) is equal to limiting friction (μR), proving the statement.

(b)

C Part 1	C Part 2	D	Extra Equations
$u_1 = 0$	$u_{C2} = u$	$u_D = \frac{3u}{4}$	~~$t_{C1} + t_{C2} = t_B$~~
$v_{C1} = u$	$v_{C2} = u$	$v_D = \frac{3u}{4}$	~~$s_{C1} + s_{C2} = d$~~
$a_{C1} = a$	$a_{C2} = 0$	$a_D = 0$	
$s_{C1} = d - s$	$s_{C2} = s$	$s_D = d$	
$t_{C1} = T - t$	$t_{C2} = t$	$t_D = T$	

We want to show that

$$t = \frac{4d}{3u} - \frac{u}{a},$$

making s, T non-equation variables (this is why I gave t_{C2} the simpler expression). Using the second column,

$$s_{C2} = u_{C2}t_{C2}$$
$$\Rightarrow s = ut.$$

Using the first column,

$$\begin{aligned} v_{C1}^2 &= u_{C1}^2 + 2a_{C1}s_{C1} \\ \Rightarrow u^2 &= 2a(d - ut) \\ \Rightarrow \frac{u^2}{2a} &= d - ut \\ \Rightarrow ut &= d - \frac{u^2}{2a} \\ \Rightarrow t &= \frac{d}{u} - \frac{u}{2a}. \end{aligned}$$

(c) We now want d, u and a in an equation, making s, t and T non-equation variables. We know

$$t = \frac{d}{u} - \frac{u}{2a},$$

so that

$$\begin{aligned} s &= ut \\ &= d - \frac{u^2}{2a}. \end{aligned}$$

Using the third column,

$$\begin{aligned} s_D &= u_D t_D \\ \Rightarrow d &= \frac{3Tu}{4} \\ \Rightarrow \frac{4d}{3u} &= T. \end{aligned}$$

Our UVAST array now looks like this.

C Part 1	C Part 2	D
$u_{C1} = 0$	$u_{C2} = u$	$u_D = \frac{3u}{4}$
$v_{C1} = u$	$v_{C2} = u$	$v_D = \frac{3u}{4}$
$a_{C1} = a$	$a_{C2} = 0$	$a_D = 0$
$s_{C1} = \cancel{d - s}\ \frac{u^2}{2a}$	$s_{C2} = \cancel{s}\ d - \frac{u^2}{2a}$	$s_D = d$
$t_{C1} = \cancel{T - t}\ \frac{d}{3u} + \frac{u}{2a}$	$t_{C2} = \cancel{t}\ \frac{d}{u} - \frac{u}{2a}$	$t_D = \cancel{T}\ \frac{4d}{3u}$

We've used one equation in each column. We can only use one column in the second and third (as $a = 0$) but two in the first (as it's full of numbers/variables and $a \neq 0$). It doesn't matter what the second equation is, so making the algebra easier

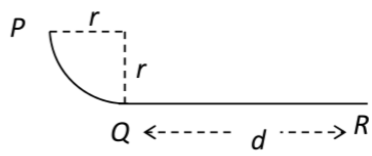
$$
\begin{aligned}
v_{C1} &= u_{C1} + a_{C1}t_{C1} \\
\Rightarrow u &= a\left(\frac{d}{3u} + \frac{u}{2a}\right) \\
\Rightarrow \frac{u}{a} &= \frac{d}{3u} + \frac{u}{2a} \\
\Rightarrow \frac{u}{2a} &= \frac{d}{3u} \\
\Rightarrow \frac{3u^2}{2a} &= d.
\end{aligned}
$$

Example — 2017 Q1.

(a) A car passes four collinear markers A, B, C, and D while moving in a straight line with uniform acceleration. The car takes t seconds to travel from A to B, t seconds to travel from B to C and t seconds to travel from C to D.

If $|AB| + |CD| = k|BC|$, find the value of k.

(b) A baggage chute has two sections, PQ and QR, as shown in the diagram.
PQ is smooth and is a quarter circle of radius r.
QR, of length d, is rough and horizontal.
The coefficient of friction between the bag and section QR is μ.



A bag of mass m kg is released from rest at P and comes to rest at R.

Find

(i) the speed of the bag at Q in terms of r

(ii) d in terms of μ and r.

The speed of the bag when it is halfway along QR is 7 m s^{-1}.

(iii) Find the value of r.

(a)

AB	BC	CD	Extra Equations
$u_1 =$	$u_2 = v$	$u_3 = w$	~~$v_1 = u_2$~~
$v_1 = v$	$v_2 = w$	$v_3 =$	~~$v_2 = u_3$~~
$a_1 = a$	$a_2 = a$	$a_3 = a$	~~$a_1 = a_2 = a_3$~~
$s_1 = \|AB\|$	$s_2 = \|BC\|$	$s_3 = \|CD\|$	
$t_1 = t$	$t_2 = t$	$t_3 = t$	

We want to get s_1, s_2, s_3 in terms of other variables, ideally the same variables so they're comparable.

$$s_1 = v_1 t_1 - \frac{1}{2} a_1 t_1^2$$
$$\Rightarrow |AB| = vt - \frac{1}{2} at^2.$$

$$s_2 = u_2t_2 + \frac{1}{2}a_1t_1^2$$
$$\Rightarrow |BC| = vt + \frac{1}{2}at^2.$$

If $|AB|$ and $|BC|$ are in terms of v and t, we don't want $|CD|$ in terms of w. We can use the second column to get w in terms of v.

$$v_2 = u_2 + a_2t_2$$
$$\Rightarrow w = v + at.$$

Then

$$\begin{aligned} s_3 &= u_3t_3 + \frac{1}{2}a_3t_3^2 \\ \Rightarrow |CD| &= (v+at)t + \frac{1}{2}at^2 \\ &= vt + at^2 + \frac{1}{2}at^2 \\ &= vt + \frac{3}{2}at^2. \end{aligned}$$

Then

$$\begin{aligned} |AB| + |CD| &= vt - \frac{1}{2}at^2 + vt + \frac{3}{2}at^2 \\ &= 2vt + at^2 \\ &= 2(vt + \frac{1}{2}at^2) \\ &= 2|BC|, \end{aligned}$$

so that $k = 2$.

(b) (i) Applying the Principle of Conservation of Energy to compare the energies at P and Q,

$$\begin{aligned} \text{P.E.}_{\text{Before}} + \text{K.E.}_{\text{Before}} &= \text{P.E.}_{\text{After}} + \text{K.E.}_{\text{After}} \\ \Rightarrow mgr + 0 &= 0 + \frac{mv^2}{2} \\ \Rightarrow 2gr &= v^2 \\ \Rightarrow \sqrt{2gr} &= v. \end{aligned}$$

(ii) As the particle slides along QR the following forces are acting on the particle.

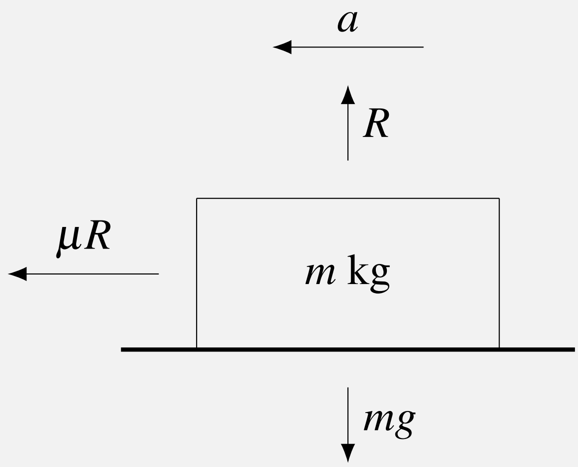


Figure 6

We quickly have, as usual, $R = mg$. Then

$$\begin{aligned} \mu R &= ma \\ \Rightarrow \mu mg &= ma \\ \Rightarrow \mu g &= a. \end{aligned}$$

If the particle comes to rest at R then we have the following UVAST array for the journey QR.

$$\begin{aligned} u &= \sqrt{2gr} \\ v &= 0 \\ a &= -\mu g \\ s &= d \\ t &= \end{aligned}$$

Then

$$\begin{aligned} v^2 &= u^2 + 2as \\ \Rightarrow 0 &= \sqrt{2gr}^2 + 2(-\mu g)d \\ \Rightarrow 2\mu dg &= 2gr \\ \Rightarrow d &= \frac{r}{\mu}. \end{aligned}$$

(iii) In this case we have the following UVAST array to the halfway point.

$$\begin{aligned} u &= \sqrt{2gr} \\ v &= 7 \\ a &= -\mu g \\ s &= \frac{r}{2\mu} \\ t &= \end{aligned}$$

Then

$$
\begin{aligned}
v^2 &= u^2 + 2as \\
\Rightarrow 7^2 &= \sqrt{2gr}^2 + 2(-\mu g)\left(\frac{r}{2\mu}\right) \\
\Rightarrow 49 &= 2gr - gr \\
\Rightarrow 49 &= gr \\
\Rightarrow 5\text{ m} &= r.
\end{aligned}
$$

Example — 2016 Q1.

(a) A car has an initial speed of u m s^{-1}. It moves in a straight line with constant acceleration f for 4 seconds. It travels 40 m while accelerating.
The car then moves with uniform speed and travels 45 m in 3 seconds.
It is then brought to rest by a constant retardation $2f$.

(i) Draw a speed-time graph for the motion.

(ii) Find the value of u.

(iii) Find the total distance travelled.

(b) A particle is projected vertically upwards with a velocity of u m s^{-1}.
After an interval of $2t$ seconds a second particle is projected vertically upwards from the same point and with the same initial velocity.

They meet at a height of h m.

Show that $h = \dfrac{u^2 - g^2t^2}{2g}$.

(a) (i)

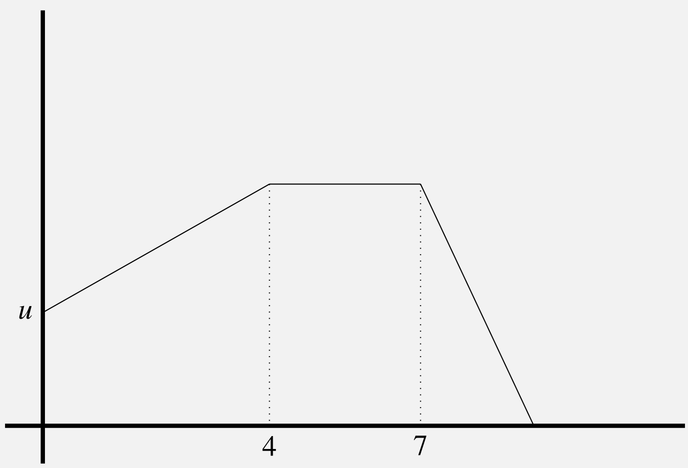


Figure 7

Remember that calculations are unnecessary if asked to draw a speed time graph at the beginning of the question. Minimal calculations give a speed of 15 m/s for the middle period but it is not required to get full points according to the marking scheme.

(ii)

First Part	Second Part	Third Part	Extra Equations
$u_1 = u$	$u_2 = v$	$u_3 = v$	~~$u_2 = v_1 = v_2 = u_3$~~
$v_1 = v$	$v_2 = v$	$v_3 = 0$	
$a_1 = f$	$a_2 = 0$	$a_3 = -2f$	
$s_1 = 40$	$s_2 = 45$	$s_3 =$	
$t_1 = 4$	$t_2 = 3$	$t_3 =$	

$$\begin{aligned} s_2 &= u_2 t_2 \\ \Rightarrow 45 &= 3t \\ \Rightarrow 15 &= v. \end{aligned}$$

Then

$$\begin{aligned} s_1 &= \left(\frac{u_1 + v_1}{2}\right) t_1 \\ \Rightarrow 40 &= \frac{u+15}{2}(4) \\ &= 2u + 30 \\ \Rightarrow 10 &= 2u \\ \Rightarrow 5 &= u. \end{aligned}$$

(iii)

$$\begin{aligned} v_1 &= u_1 + a_1 t_1 \\ \Rightarrow 15 &= 5 + 4f \\ \Rightarrow 2.5 &= f. \end{aligned}$$

Then

$$\begin{aligned} v_3^2 &= u_3^2 + 2a_3 s_3 \\ \Rightarrow 0 &= 15^2 - 10s \\ \Rightarrow 10s &= 225 \\ \Rightarrow s &= 22.5 \end{aligned}$$

so that the total distance travelled is

$$40 + 45 + 22.5 = 107.5 \text{ m}.$$

Example — 2015 Q1.

Note: One could argue that part (b) of this question relies on relative acceleration and so is not relevant to the new syllabus. However the solution below does not use any concepts from relative velocity.

(a) A particle starts from rest and moves with constant acceleration.

If the particle travels 39 m in the seventh second, find the distance travelled in the tenth second.

(b) A train of length 66.5 m is travelling with uniform acceleration $\frac{4}{7}$ m s^{-2}.

It meets another train of length 91 m travelling on a parallel track in the opposite direction with uniform acceleration $\frac{8}{7}$ m s^{-2}.

Their speeds at this moment are 18 m s^{-1} and 24 m s^{-1} respectively.

(i) Find the time taken for the trains to pass each other.

(ii) Find the distance between the trains 1 second later.

(a)

First Part	Second Part	Third Part	Fourth Part	Extra Equations
$u_1 = 0$	$u_2 = v$	$u_3 = w$	$u_4 = x$	~~$u_2 = v_1$~~
$v_1 = v$	$v_2 = w$	$v_3 = x$	$v_4 =$	~~$u_3 = v_2$~~
$a_1 = a$	$a_2 = a$	$a_3 = a$	$a_4 = a$	~~$u_4 = v_3$~~
$s_1 =$	$s_2 = 39$	$s_3 =$	$s_4 =$	~~$a_1 = a_2 = a_3 = a_4$~~
$t_1 = 6$	$t_2 = 1$	$t_3 = 2$	$t_4 = 1$	

$$\begin{aligned} v_1 &= u_1 + a_1 t_1 \\ \Rightarrow v &= 6a. \end{aligned}$$

$$\begin{aligned} v_2 &= u_2 + a_2 t_2 \\ \Rightarrow w &= 6a + a \\ &= 7a. \end{aligned}$$

$$\begin{aligned} s_2 &= \left(\frac{u_2 + v_2}{2}\right) t_2 \\ \Rightarrow 39 &= \frac{6a + 7a}{2} \\ \Rightarrow 78 &= 13a \\ \Rightarrow 6 &= a. \end{aligned}$$

As an update, our UVAST array now looks like this.

First Part	Second Part	Third Part	Fourth Part	Extra Equations
$u_1 = 0$	$u_2 = \cancel{v}\,36$	$u_3 = \cancel{w}\,42$	$u_4 = x$	$\cancel{u_2 = v_1}$
$v_1 = \cancel{v}\,36$	$v_2 = \cancel{w}\,42$	$v_3 = x$	$v_4 =$	$\cancel{u_3 = v_2}$
$a_1 = \cancel{a}\,6$	$a_2 = \cancel{a}\,6$	$a_3 = \cancel{a}\,6$	$a_4 = \cancel{a}\,6$	$\cancel{u_4 = v_3}$
$s_1 =$	$s_2 = 39$	$s_3 =$	$s_4 =$	$\cancel{a_1 = a_2 = a_3 = a_4}$
$t_1 = 6$	$t_2 = 1$	$t_3 = 2$	$t_4 = 1$	

Then

$$\begin{aligned} v_3 &= u_3 + a_3 t_3 \\ \Rightarrow x &= 42 + 6(2) \\ &= 54. \end{aligned}$$

Finally

$$\begin{aligned} s_4 &= u_4 t_4 + \frac{1}{2} a_4 t_4^2 \\ &= 54 + 3 \\ &= 57 \text{ m}. \end{aligned}$$

(b) (i) If we consider the journey of the noses of the trains, from when they pass each other to when the tails of the train meet, the noses travel a total of $91 + 66.5 = 157.5$ m, the combined lenght of the trains and so we have the following UVAST array.

Train A	Train B	Extra Equations
$u_A = 18$	$u_B = 24$	$\cancel{t_A = t_B}$
$v_A =$	$v_B =$	$\cancel{s_A + s_B = 157.5}$
$a_A = \frac{4}{7}$	$a_B = \frac{8}{7}$	
$s_A = s$	$s_B = 157.5 - s$	
$t_A = t$	$t_B = t$	

$$\begin{aligned} s_A &= u_A t_A + \frac{1}{2} a_A t_A^2 \\ \Rightarrow s &= 18t + \frac{2}{7} t^2. \end{aligned}$$

$$\begin{aligned} s_B &= u_B t_B + \frac{1}{2} a_B t_B^2 \\ \Rightarrow 157.5 - s &= 24t + \frac{4}{7} t^2 \\ \Rightarrow 157.5 - \left(18t + \frac{2}{7} t^2\right) &= 24t + \frac{4}{7} t^2 \\ \Rightarrow 0 &= \frac{6}{7} t^2 + 42t - 157.5 \\ \Rightarrow t &= \cancel{-52.5}, 3.5 \text{ seconds}. \end{aligned}$$

(ii) Creating a UVAST array for the first $3.5+1=4.5$ seconds of the journey from when the noses meet,

Train A	Train B
$u_A = 18$	$u_B = 24$
$v_A =$	$v_B =$
$a_A = \frac{4}{7}$	$a_B = \frac{8}{7}$
$s_A =$	$s_B =$
$t_A = 4.5$	$t_B = 4.5$

$$\begin{aligned} s_A &= u_A t_A + \frac{1}{2} a_A t_A^2 \\ &= 18(4.5) + \frac{2}{7}(4.5)^2 \\ &= \frac{1215}{14}. \end{aligned}$$

$$\begin{aligned} s_B &= u_B t_B + \frac{1}{2} a_B t_B^2 \\ &= 24(4.5) + \frac{4}{7}(4.5)^2 \\ &= \frac{837}{7}. \end{aligned}$$

Then the distance between the tails of the trains is

$$\frac{1215}{14} + \frac{837}{7} - 157.5 = \frac{342}{7} \text{ m.}$$