Differential Equations Exam Question Solutions

Note These exam questions are given in reverse chronological order as they appear in exam papers; 2023 (deferred), 2023, Sample paper, 2022 (deferred), 2022, and so on back to 2015. As all old syllabus questions are relevant to the new syllabus all are included. Some exam questions that are related to earlier questions involving difference equations are also included, so some understanding of difference equations would be required.

In the solving of initial value problems, sometimes definite integrals are used and sometimes indefinite integrals are used. This is based on the perceived speed of each method for the question at hand, although both can be used for any initial value problem. As a rule of thumb, indefinite integrals are used if integration results in logarithms and definite integrals are used otherwise.

Note that these solutions calculate anti-derivatives of functions of the form $f(a+bx)$ where $f(x)$ is a simple function whose integral is known, without stating that integration by substitution is being applied. For example, integrals like $\int \frac{1}{3+2x}\,dx = \frac{1}{2}\ln|3+2x| + C$, or $\int \cos(5-2t)\,dt = \frac{1}{-2}\sin(5-2t) + C$ are done in one line, as shown in my notes.

Question — 2023 Deferred Q2 (b)(iii).

(b) In another, larger computer network, a message travels from one computer to another in the network. Each time a message travels from one computer to the next the number of errors in the message, E, increases by 15%. However C errors are corrected each time the message travels. The number of computers the message travels to is counted using the number n.

A message starts at computer $n = 0$ and travels on a linear path through the computer network.

E, the number of errors in the message, may be modelled by the difference equation:

$$E_{n+1} = 1.15E_n - C$$

where $n \geq 0, n \in \mathbb{Z}$.

There are 101 errors in the message when it leaves computer 0, i.e. $E_0 = 101$.

(iii) E may also be modelled using a differential equation. Write a differential equation for $\frac{dE}{dn}$, the rate of change of E with respect to n, in terms of E and C.

The idea is to replace $E_{n+1} - E_n$, the discrete difference, with $\frac{dE}{dn}$, the continuous rate of change. Then replace E_n with E.

$$\begin{aligned} E_{n+1} &= 1.15E_n - C \\ \Rightarrow E_{n+1} - E_n &= 0.15E_n - C \\ \Rightarrow \frac{dE}{dn} &= 0.15E - C. \end{aligned}$$

Question — 2023 Deferred Q4.

Question 4

In 1838 the Belgian mathematician Pierre François Verhulst published a differential equation to model rate of change of population P with respect to time t:

$$\frac{dP}{dt} = rP\left(1 - \frac{P}{K}\right)$$

where r and K are constants for a given population.

For a certain species of insect in an environment it is known that the population can increase by up to 8% per week, i.e. $r = 0.08$.

At $t = 0$ weeks there are 20 insects in the population.

When the population P is small relative to K, the ratio $\frac{P}{K}$ is also small and Verhulst's model can be approximated by the simplified differential equation:

$$\frac{dP}{dt} = rP$$

(i) Solve this simplified differential equation to find an expression for P in terms of t.

(ii) Calculate P to the nearest whole number when $t = 12$ weeks.

(iii) Explain why this approximation of Verhulst's model is not practical for predicting the long-term behaviour of the population of insects.

(iv) Solve the differential equation for Verhulst's model:

$$\frac{dP}{dt} = rP\left(1 - \frac{P}{K}\right)$$

to find an expression that relates P, K and t.

Note that $\frac{1}{y(x-y)} = \frac{1}{x}\left(\frac{1}{y} + \frac{1}{x-y}\right)$.

(v) $P = 39$ insects when $t = 12$ weeks. Calculate the value of K to the nearest whole number.

(vi) Explain the significance of K in the Verhulst model.

(*i*)

$$
\begin{aligned}
\frac{dP}{dt} &= 0.08P, \quad P(0) = 20 \\
\Rightarrow \int \frac{1}{P}\, dP &= \int 0.08t\, dt \\
\Rightarrow \ln|P| &= 0.08t + C \\
\Rightarrow |P| &= e^{0.08t+C} \\
\Rightarrow P &= \pm e^{C} e^{0.08t} \\
&= Ce^{0.08t}. \\
P(0) &= 20 \\
\Rightarrow 20 &= Ce^{0.08(0)} \\
\Rightarrow 20 &= C \\
\Rightarrow P &= 20e^{0.08t}.
\end{aligned}
$$

(*ii*)

$$
\begin{aligned}
P(12) &= 20e^{0.08(12)} \\
&\approx 52
\end{aligned}
$$

(*iii*) Although it's true at the beginning, P will not stay small relative to K forever and so the assumption of $1 - \frac{P}{K} \approx 1$ breaks down at some point.

(*iv*)

$$
\begin{aligned}
\frac{dP}{dt} &= 0.08P\left(1 - \frac{P}{K}\right) \\
&= 0.08P\left(\frac{K}{K} - \frac{P}{K}\right) \\
&= 0.08P\left(\frac{K-P}{K}\right) \\
\Rightarrow \int \frac{1}{P(K-P)}\, dP &= \int \frac{0.08}{K}\, dt \\
\Rightarrow \frac{1}{K}\int \frac{1}{P} + \frac{1}{K-P}\, dP &= \int \frac{0.08}{K}\, dt \\
\Rightarrow \frac{1}{K}\left(\ln|P| - \ln|K-P|\right) &= \frac{0.08}{K}t + C \\
\Rightarrow \ln\left|\frac{P}{K-P}\right| &= 0.08t + C \\
\Rightarrow \left|\frac{P}{K-P}\right| &= e^{0.08t+C} \\
\Rightarrow \frac{P}{K-P} &= \pm e^{C} e^{0.08t} \\
&= Ce^{0.08t}.
\end{aligned}
$$

C will get split up over different terms if we go much further, so now is a good time to calculate C. If $P(0) = 20$,

$$\begin{aligned}
\frac{20}{K-20} &= Ce^{0.08(0)} \\
\Rightarrow \frac{20}{K-20} &= C \\
\Rightarrow \frac{P}{K-P} &= \frac{20}{K-20}e^{0.08t} \\
\Rightarrow P &= (K-P)\frac{20}{K-20}e^{0.08t} \\
&= K\frac{20}{K-20}e^{0.08t} - P\frac{20}{K-20}e^{0.08t} \\
\Rightarrow P\left(1+\frac{20}{K-20}e^{0.08t}\right) &= K\frac{20}{K-20}e^{0.08t} \\
\Rightarrow P\left(K-20+20e^{0.08t}\right) &= 20Ke^{0.08t} \\
\Rightarrow P &= \frac{20Ke^{0.08t}}{K-20+20e^{0.08t}}.
\end{aligned}$$

(*v*) If $P(12) = 39$,

$$\begin{aligned}
39 &= \frac{20Ke^{0.08(12)}}{K-20+20e^{0.08(12)}} \\
&= \frac{20Ke^{0.96}}{K-20+20e^{0.96}} \\
\Rightarrow 39(K-20+20e^{0.96}) &= 20Ke^{0.96} \\
\Rightarrow 39K-780+780e^{0.96} &= 20Ke^{0.96} \\
\Rightarrow K(39-20e^{0.96}) &= 780-780e^{0.96} \\
\Rightarrow K &= \frac{780-780e^{0.96}}{39-20e^{0.96}} \\
\Rightarrow K &\approx 95.
\end{aligned}$$

(*vi*) K is a fixed point of the differential equation, i.e. $P(t) = K$ is a constant solution of it. The population therefore cannot exceed K, although it gets closer to it over time.

Question — 2023 Deferred Q5.

A more sophisticated model for the motion of a ball that is thrown vertically upwards includes the effects of air resistance. The rate of change of the velocity v of the ball in terms of time t during the upward part of its journey can be modelled by the following differential equation:

$$\frac{dv}{dt} = -g - kv$$

where $k > 0$ is a constant. Take the initial upward velocity of the ball to be 20 m s^{-1}.

(ii) Solve this differential equation to find an expression for v in terms of t and k.

(iii) Using $k = 0.1225$, calculate the time the ball takes to reach its maximum height.

(iv) Write down a differential equation for the rate of change of the velocity of the ball on the downward part of its journey.

(ii)

$$\begin{aligned}
\frac{dv}{dt} &= -g - kv \\
\Rightarrow \int \frac{1}{-g-kv}\, dv &= \int dt \\
\Rightarrow \frac{1}{-k} \ln|-g-kv| &= t + C \\
\Rightarrow \ln|g+kv| &= -kt + C \\
\Rightarrow |g+kv| &= e^{-kt+C} \\
\Rightarrow g + kv &= \pm e^{C} e^{-kt} \\
\Rightarrow kv &= -g + Ce^{-kt} \\
\Rightarrow v &= -\frac{g}{k} + Ce^{-kt}.
\end{aligned}$$

If $v(0) = 20$,

$$\begin{aligned}
20 &= -\frac{g}{k} + Ce^{-k(0)} \\
\Rightarrow 20 + \frac{g}{k} &= C \\
\Rightarrow v &= -\frac{g}{k} + \left(20 + \frac{g}{k}\right) e^{-kt}.
\end{aligned}$$

(*iii*) If $k = 0.1225$ and we want t when $v = 0$, putting both of these in we get

$$\begin{aligned} 0 &= -\frac{g}{0.1225} + \left(20 + \frac{g}{0.1225}\right) e^{-0.1225t} \\ \Rightarrow 80 &= 100e^{-0.1225t} \\ \Rightarrow \frac{4}{5} &= e^{-0.1225t} \\ \Rightarrow \ln\frac{4}{5} &= -0.1225t \\ \Rightarrow \frac{1}{-0.1225}\ln\frac{4}{5} &= t \\ \Rightarrow 1.82 &= t. \end{aligned}$$

(*iv*) If we take the downwards direction to be positive, then gravity acts in the same direction as acceleration but air resistance doesn't. So we get

$$\frac{dv}{dt} = g - kv.$$

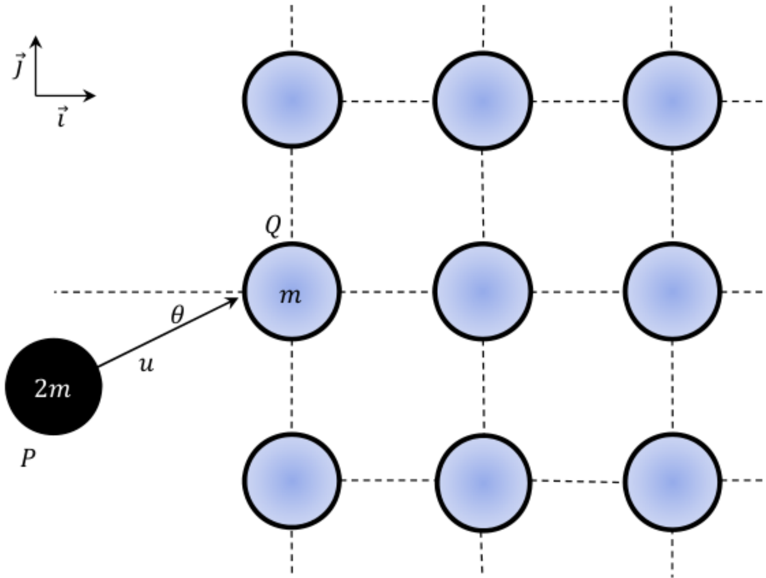
Question — 2023 Deferred Q7.

Question 7

A solid can be modelled as a two-dimensional lattice of identical particles of mass m. A particle of the solid may be moved temporarily out of its position but it is quickly returned to that position by the forces that hold the solid together.

An incoming particle P of mass $2m$, moving with speed u, collides obliquely with particle Q which is at rest on the outer surface of the solid. The line joining the centres of the particles at the point of impact is along the $\vec{\imath}$ axis.

Before the collision, the direction of P makes an angle θ with the $\vec{\imath}$ axis.



After the collision the direction of P has turned through an angle θ, such that it now makes an angle 2θ with the $\vec{\imath}$ axis.

The coefficient of restitution for the collision is $\frac{2}{3}$.

(i) Show that $\tan\theta = \frac{1}{3}$.

(ii) Calculate the $\vec{\imath}$ and $\vec{\jmath}$ components of the velocity of Q immediately after the collision in terms of u.

After the collision Q experiences a restoring force F which is proportional to x, the displacement of Q from its initial position, where k is the constant of proportionality. (That is, the restoring force may be modelled as being equivalent to the restoring force exerted on a particle by a spring of spring constant k stretched through displacement x.)

(iii) Derive an expression for the work done when Q moves through displacement x.

(iv) Find the maximum displacement of Q from its initial position in terms of m, k and u.

Note: As this document is for differential equations questions we will skip (i) and (ii): the velocity of Q after collision is $\sqrt{\frac{10}{9}}u$ rightwards.

(*iii*) We get work done by integrating the force we're resisting over the distance we resist it, so

$$\begin{aligned} W &= \int_0^x ks\, ds \\ &= \left.\frac{ks^2}{2}\right|_0^x \\ &= \frac{kx^2}{2}. \end{aligned}$$

(*iv*) Letting x be the maximum displacement,

$$\begin{aligned} F &= ma \\ \Rightarrow -ks &= ma \\ \Rightarrow -ks &= mv\frac{dv}{ds} \\ \Rightarrow \int_0^x -ks\, ds &= \int_{\sqrt{\frac{10}{9}}u}^0 mv\, dv \\ \Rightarrow -\left.\frac{ks^2}{2}\right|_0^x &= \left.\frac{mv^2}{2}\right|_{\sqrt{\frac{10}{9}}u}^0 \\ \Rightarrow -\frac{kx^2}{2} + 0 &= 0 - \frac{m\left(\sqrt{\frac{10}{9}}u\right)^2}{2} \\ \Rightarrow kx^2 &= m\frac{10}{9}u^2 \\ \Rightarrow x^2 &= \frac{10m}{9k}u^2 \\ \Rightarrow x &= \sqrt{\frac{10m}{9k}}u. \end{aligned}$$

The Work Energy equation can also be used, setting the kinetic energy at the start equal to the work done during.

Question — 2023 Q1 (b).

(b) A particle moving along a straight line has velocity $v = \frac{ds}{dt} = 2te^{-t}, t \geq 0$.

(i) Using integration by parts or otherwise, derive an expression for $s(t)$, the displacement of the particle at any time t, given that $s(0) = 0$.

(ii) Calculate $s(3)$.

(i)

$$\begin{aligned}
\frac{ds}{dt} &= 2te^{-t} \\
\Rightarrow s(t) - s(0) &= \int_0^t 2te^{-t}\, dt. \\
u = 2t &\text{ and } dv = e^{-t}\, dt \\
\Rightarrow du = 2\, dt &\text{ and } v = -e^{-t} \\
\Rightarrow s(t) &= 2t(-e^{-t})\Big|_0^t - \int_0^t (-e^{-t})(2\, dt) \\
&= -2te^{-t}\Big|_0^t + \int_0^t 2e^{-t}\, dt \\
&= -2te^{-t} + 0 - 2e^{-t}\Big|_0^t \\
&= -2te^{-t} - 2e^{-t} + 2e^{-0} \\
&= 2 - 2te^{-t} - 2e^{-t}.
\end{aligned}$$

(ii)

$$\begin{aligned}
s(3) &= 2 - 6e^{-3} - 2e^{-3} \\
&= 2 - 8e^{-3}.
\end{aligned}$$

Question — 2023 Q4.

Question 4

A ball of mass m kg is projected with initial velocity 15 m s^{-1} vertically downwards into a tank of water. The ball travels through the water against an upward buoyancy force that is 4 times the magnitude of the weight of the ball and a drag force of mv^2 N.

(i) Draw a diagram to show the forces acting on the ball while it is moving downwards through the water.

(ii) Show that, while the ball is moving downwards, the rate of change of its velocity v with respect to its distance s below the surface of the water can be expressed by the differential equation:

$$v\frac{dv}{ds} = -29.4 - v^2$$

(iii) Solve this differential equation to find an expression for v in terms of s.

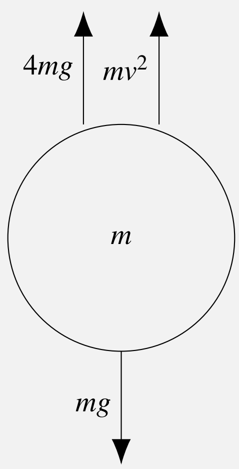
(iv) The ball is at its maximum depth, D, when $v = 0$. Calculate D.

After reaching its maximum depth the ball changes direction and begins to move upwards through the water.

(v) Draw a diagram to show the forces acting on the ball while it is moving upwards through the water.

(vi) Write down a differential equation for the rate of change of the velocity v of the ball while it moves upwards through the water.

(*i*)



(*ii*) Taking the downwards direction as positive,

$$F = ma$$
$$\Rightarrow mg - 4mg - v^2 = ma$$
$$\Rightarrow -3g - v^2 = a$$
$$\Rightarrow -29.4 - v^2 = v\frac{dv}{ds}.$$

(*iii*) Using integration by substitution on the v side,

$$
\begin{aligned}
-29.4 - v^2 &= v\frac{dv}{ds} \\
\Rightarrow \int ds &= \int \frac{v}{-29.4 - v^2}\, dv. \\
\text{Let } u &= -29.4 - v^2 \\
\Rightarrow du &= -2v\, dv \\
\Rightarrow -\frac{1}{2}\, du &= v\, dv \\
\Rightarrow \int ds &= \int \frac{1}{u}\left(-\frac{1}{2}\right) du \\
\Rightarrow s + C &= -\frac{1}{2}\ln|u| \\
\Rightarrow -2s + C &= \ln|-29.4 - v^2| \\
\Rightarrow e^{-2s+C} &= |29.4 + v^2| \\
\Rightarrow \pm e^C e^{-2s} &= 29.4 + v^2 \\
\Rightarrow Ce^{-2s} - 29.4 &= v^2.
\end{aligned}
$$

If we go any further C will be stuck inside the square root function so now is a good time to calculate C. We started at the top of the water (so $s = 0$) at a velocity of $v = +15$ (it is $+15$ as we took the downwards direction to be positive)

$$
\begin{aligned}
Ce^{-2(0)} - 29.4 &= 15^2 \\
\Rightarrow C &= 254.4.
\end{aligned}
$$

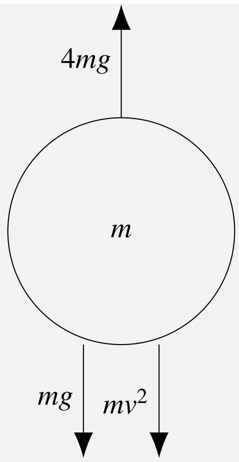
Then

$$
\begin{aligned}
254.4e^{-2s} - 29.4 &= v^2 \\
\Rightarrow \sqrt{254.4e^{-2s} - 29.4} &= v.
\end{aligned}
$$

(*iv*)

$$
\begin{aligned}
\sqrt{254.4e^{-2D} - 29.4} &= 0 \\
\Rightarrow 254.4e^{-2D} - 29.4 &= 0 \\
\Rightarrow 254.4e^{-2D} &= 29.4 \\
\Rightarrow e^{-2D} &= \frac{29.4}{254.4} \\
\Rightarrow -2D &= \ln\frac{29.4}{254.4} \\
\Rightarrow D &= 1.08 \text{ m.}
\end{aligned}
$$

(*v*) Drag acts in the opposite direction while the other forces are unchanged.



(vi) If the upwards direction is now positive,

$$
\begin{aligned}
F &= ma \\
\Rightarrow 4mg - mg - mv^2 &= ma \\
\Rightarrow 29.4 - v^2 &= v\frac{dv}{ds}.
\end{aligned}
$$

Question — 2023 Q7 (b).

(b) A *learning curve* is a graphical representation of how a person's ability to perform a certain task increases with the time the person spends learning or practicing that task.

A student wishes to be able to spell 2000 difficult words. The rate of the student's learning may be modelled by the differential equation:

$$\frac{dN}{dt} = k(2000 - N)$$

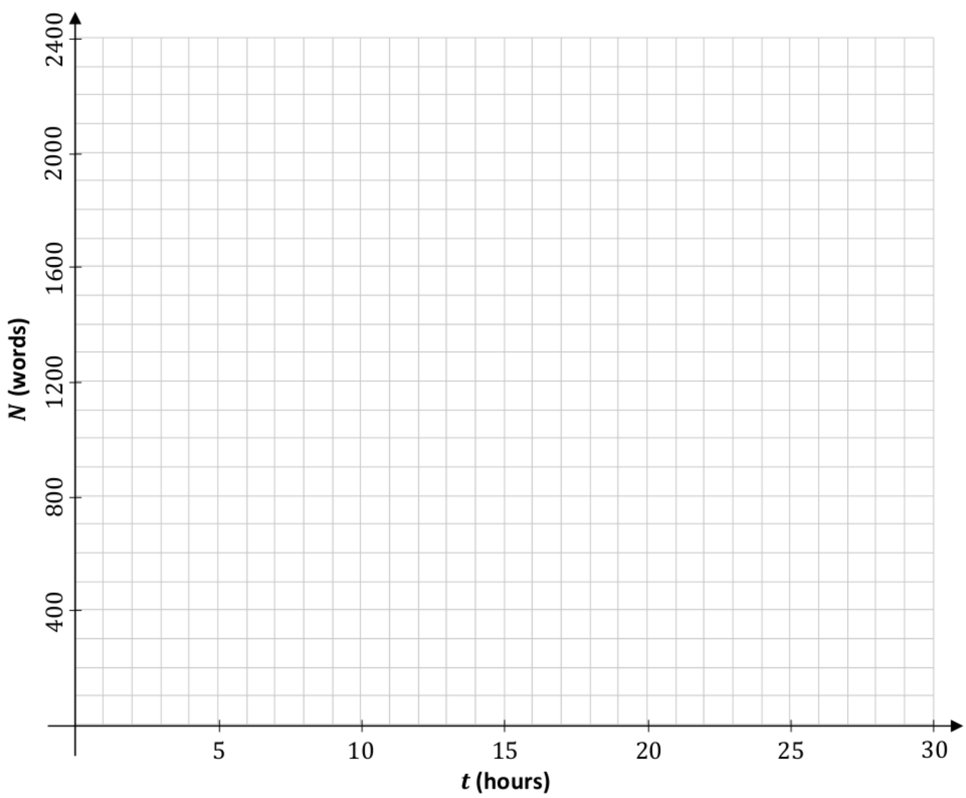
where $N(t)$ is number of these words the student is able to spell after t hours of learning, and where k is a positive constant.

At the start of their learning the student is already able to spell 250 of these words, i.e. $N(0) = 250$.

(i) Solve the differential equation to find an expression for N in terms of k and t.

(ii) After 6 hours of learning, the student is able to spell 1500 of these words. Calculate k.

(iii) Sketch the shape of a graph of N against t to show the model's prediction for the student's learning curve.



(*i*)

$$\begin{aligned}
\frac{dN}{dt} &= k(2000-N) \\
\Rightarrow \int \frac{dN}{2000-N} &= \int k\,dt \\
\Rightarrow -\ln|2000-N| &= kt+C \\
\Rightarrow \ln|2000-N| &= -kt+C \\
\Rightarrow |2000-N| &= e^{-kt+C} \\
\Rightarrow 2000-N &= \pm e^{C}e^{-kt} \\
\Rightarrow 2000-N &= Ce^{-kt}.
\end{aligned}$$

If $N(0)=250$,

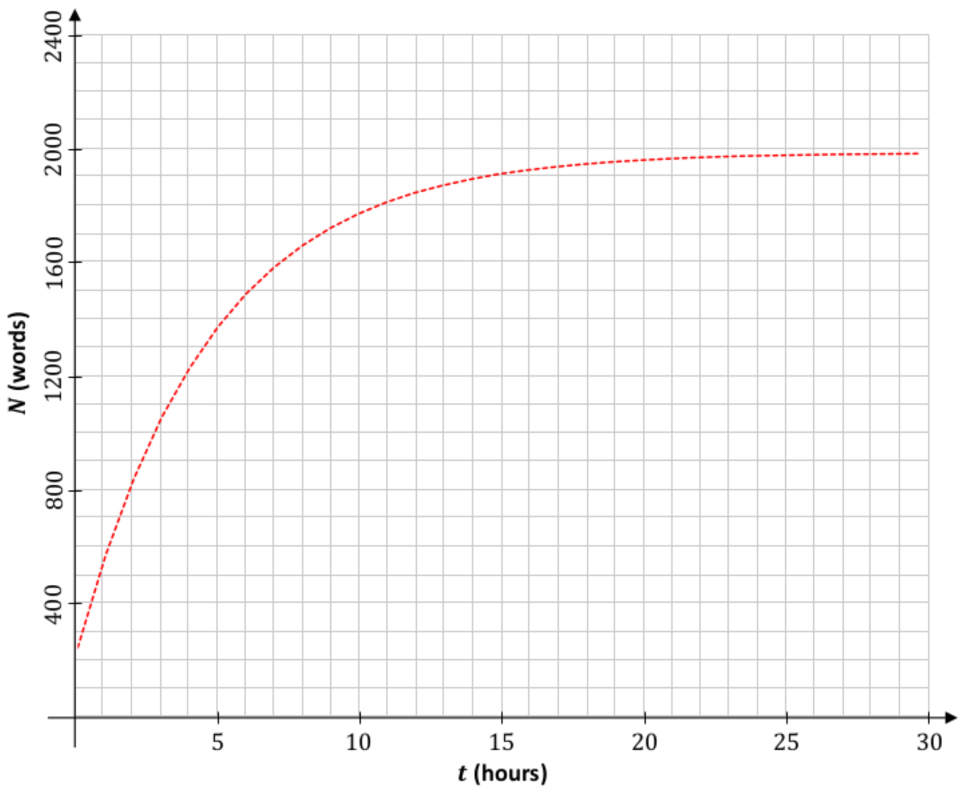
$$\begin{aligned}
2000-250 &= Ce^{-k(0)} \\
\Rightarrow 1750 &= C \\
\Rightarrow 2000-N &= 1750e^{-kt} \\
\Rightarrow 2000-1750e^{-kt} &= N.
\end{aligned}$$

(*ii*) If $N(6)=1500$,

$$\begin{aligned}
2000-1750e^{-k(6)} &= 1500 \\
\Rightarrow 500 &= 1750e^{-6k} \\
\Rightarrow \frac{2}{7} &= e^{-6k} \\
\Rightarrow \ln\frac{2}{7} &= -6k \\
\Rightarrow \frac{\ln\frac{2}{7}}{-6} &= k \\
\Rightarrow 0.209 &= k.
\end{aligned}$$

(*iii*) If $N(t)=2000-1750e^{-0.209t}$ then the graph starts at 250 when $t=0$ and tends to 2000 as time increases.

7(b) (iii)



Question — Sample Q3 (a).

Question 3

(a) A particle has initial displacement s_0 from a fixed point P. It moves away from P with initial velocity u and constant acceleration $a = \frac{dv}{dt} = \frac{d^2s}{dt^2}$.

Use calculus to derive an expression for s, the displacement of the particle from P at any time t.

$$
\begin{aligned}
\frac{dv}{dt} &= a \\
\Rightarrow v(t) &= \int a \, dt \\
&= at + C. \\
v(0) &= u \\
\Rightarrow u &= a(0) + C \\
\Rightarrow u &= C \\
\Rightarrow v &= at + u \\
\Rightarrow \frac{ds}{dt} &= at + u \\
\Rightarrow s &= \int (at + u) \, dt \\
&= \frac{at^2}{2} + ut + C. \\
s(0) &= s_0 \\
\Rightarrow s_0 &= \frac{a(0)^2}{2} + u(0) + C \\
\Rightarrow s_0 &= C \\
\Rightarrow s &= \frac{at^2}{2} + ut + s_0.
\end{aligned}
$$

Question — Sample Q5 (b).

(b) A rumour may be spread when a person who has heard the rumour interacts with a person who has not heard the rumour.

Therefore, the rate of spread of a rumour within a group can be modelled as being proportional to the product of the number of people in the group who have heard the rumour and the number of people in the group who have not heard it.

A student models the rate at which a certain rumour spreads within a school population of 1200 students using the differential equation:

$$\frac{dR}{dt} = kR(1200 - R)$$

where $R(t)$ is the number of students of that school who have heard the rumour at time t, measured in days, and where k is a positive constant.

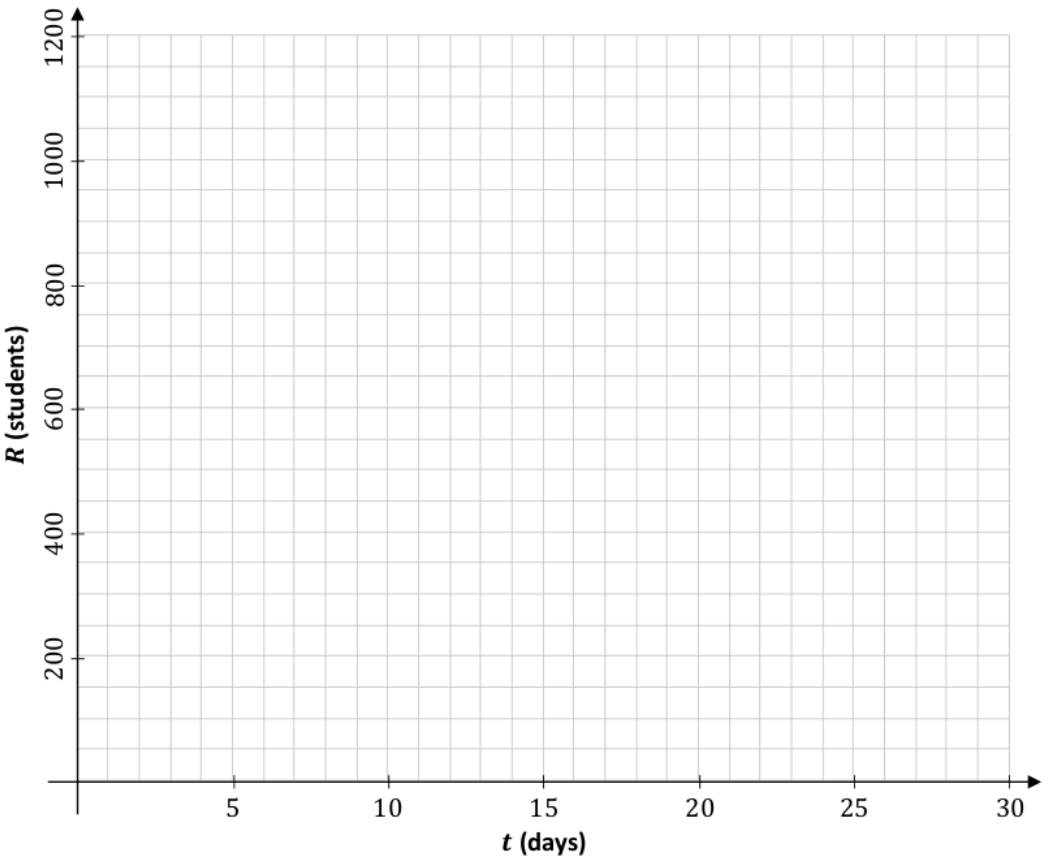
On Monday morning ($t = 0$), 100 students had heard the rumour.

(i) Solve the differential equation to find an expression that relates R, k and t.

Note that $\frac{1}{y(x-y)} = \frac{1}{x}\left(\frac{1}{y} + \frac{1}{x-y}\right)$.

(ii) By Wednesday morning 250 students had heard the rumour. Calculate the value of k.

(iii) Sketch the shape of a graph of R against t to show how the model predicts the spread of the rumour.



(i)

$$
\begin{aligned}
\frac{dR}{dt} &= kR(1200-R) \\
\Rightarrow \int \frac{1}{R(1200-R)}\,dR &= \int k\,dt \\
\Rightarrow \frac{1}{1200}\int \frac{1}{R}+\frac{1}{1200-R}\,dR &= \int k\,dt \\
\Rightarrow \frac{1}{1200}\left(\ln|R|-\ln|1200-R|\right) &= kt+C \\
\Rightarrow \ln\frac{|R|}{|1200-R|} &= 1200kt+C \\
\Rightarrow \left|\frac{R}{1200-R}\right| &= e^{1200kt+C} \\
\Rightarrow \frac{R}{1200-R} &= \pm e^{C}e^{1200kt} \\
&= Ce^{1200kt}.
\end{aligned}
$$

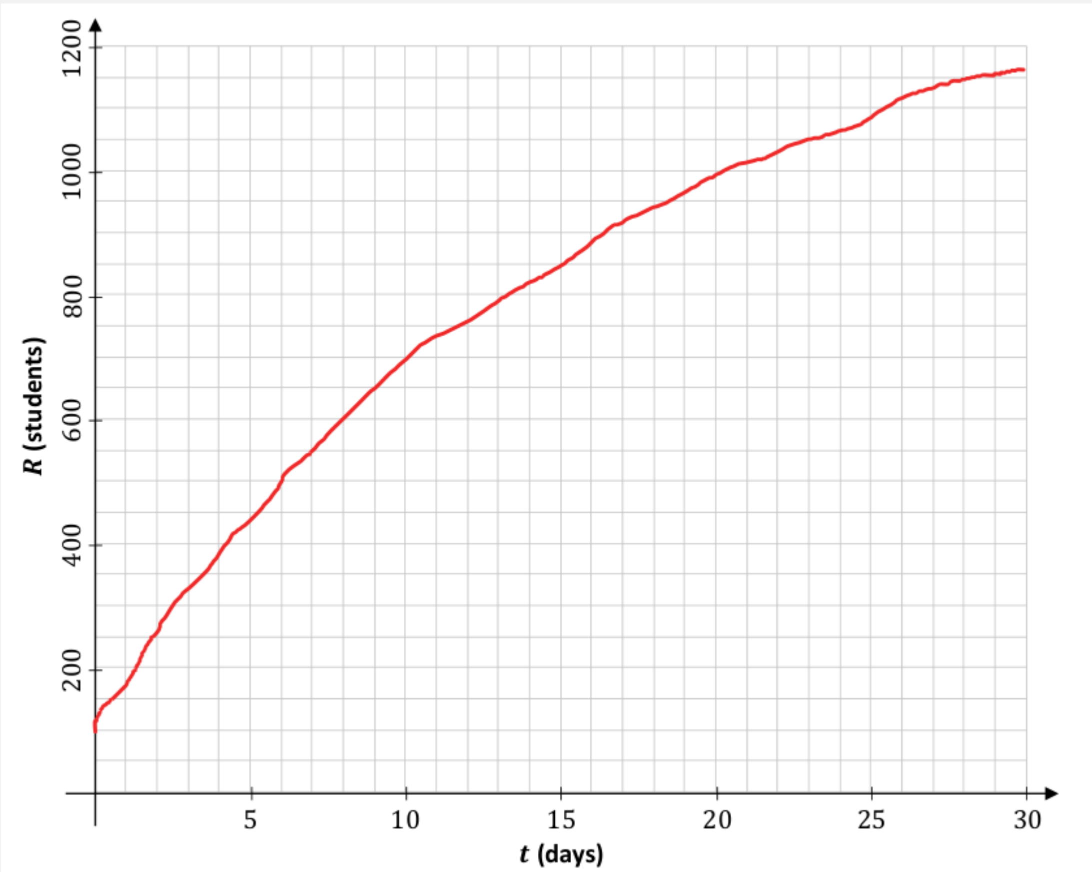
This is a good time to find C. If $R(0)=100$,

$$
\begin{aligned}
\frac{100}{1200-100} &= Ce^{1200k(0)} \\
\Rightarrow \frac{1}{11} &= C \\
\Rightarrow \frac{R}{1200-R} &= \frac{1}{11}e^{1200kt} \\
\Rightarrow R &= (1200-R)\frac{1}{11}e^{1200kt} \\
&= \frac{1200}{11}e^{1200kt}-\frac{R}{11}e^{1200kt} \\
\Rightarrow R\left(1+\frac{1}{11}e^{1200kt}\right) &= \frac{1200}{11}e^{1200kt} \\
\Rightarrow R &= \frac{\frac{1200}{11}e^{1200kt}}{1+\frac{1}{11}e^{1200kt}} \\
&= \frac{1200e^{1200kt}}{11+e^{1200kt}}.
\end{aligned}
$$

(*ii*) Going back to a previous equation, if $R(2) = 250$ then

$$\begin{aligned}
\frac{R}{1200-R} &= \frac{1}{11}e^{1200kt} \\
\Rightarrow \frac{250}{1200-250} &= \frac{1}{11}e^{1200k(2)} \\
\Rightarrow \frac{5}{19} &= \frac{1}{11}e^{2400k} \\
\Rightarrow \frac{55}{19} &= e^{2400k} \\
\Rightarrow \ln\frac{55}{19} &= 2400k \\
\Rightarrow \frac{1}{2400}\ln\frac{55}{19} &= k \\
\Rightarrow 0.00044 &= k.
\end{aligned}$$

(*iii*) The important properties of the graph is that it passes through $(0,100)$, $(2,250)$ and approaches (but never quite reaches) 1200 as $t \to \infty$.



Question — Sample Q7 (a)(i).

Question 7

(a) A bungee jumper of mass 75 kg jumps from a height of 35 m above water. The jumper is tied to an elastic rope of natural length 12 m and elastic constant 100 N m^{-1}.

(i) Derive an expression for the work done when a spring of elastic constant k N m^{-1} is stretched by x m.

The force being resisted when stretching the spring is ks where s is the distance the spring has already been stretched, and so

$$
\begin{aligned}
W &= \int_0^x ks\, ds \\
&= \left.\frac{ks^2}{2}\right|_0^x \\
&= \frac{kx^2}{2}.
\end{aligned}
$$

Question — 2022 Deferred Q10.

10. **(a)** **(i)** Solve the differential equation

$$(1+t^2)\frac{dr}{dt} = 1$$

given that $r = 0$ when $t = \frac{\pi}{4}$.

(ii) If

$$\frac{dy}{dx} = (y+4)\cos^2 3x$$

and $y = -3$ when $x = 0$, find the value of y when $x = \frac{\pi}{6}$.

(b) A particle is projected horizontally along a smooth horizontal surface with initial speed 80 m s^{-1}. The particle has a retardation of $\frac{v}{100}$ m s^{-2}, where v is the speed.

Find

(i) the speed of the particle after t seconds

(ii) the distance travelled in t seconds

(iii) the speed v in terms of the distance travelled, s.

(a) *(i)*

$$\begin{aligned}
(1+t^2)\frac{dr}{dt} &= 1 \\
\Rightarrow \int_0^r dr &= \int_{\frac{\pi}{4}}^t \frac{1}{1+t^2}\,dt \\
\Rightarrow r\Big|_0^r &= \tan^{-1} t\Big|_{\frac{\pi}{4}}^t \\
\Rightarrow r - 0 &= \tan^{-1} t - \tan^{-1}\frac{\pi}{4} \\
\Rightarrow r &= \tan^{-1} t - 0.67.
\end{aligned}$$

(ii)

$$\begin{aligned}
\frac{dy}{dx} &= (y+4)\cos^2 3x \\
\Rightarrow \int \frac{1}{y+4}\,dy &= \int \cos^2 3x\,dx \\
\Rightarrow \int \frac{1}{y+4}\,dy &= \int \frac{1}{2}(1+\cos 6x)\,dx \\
\Rightarrow \ln|y+4| &= \frac{x}{2} + \frac{1}{12}\sin 6x + C \\
\Rightarrow |y+4| &= e^{\frac{x}{2}+\frac{1}{12}\sin 6x + C} \\
\Rightarrow y+4 &= \pm e^C e^{\frac{x}{2}+\frac{1}{12}\sin 6x} \\
&= Ce^{\frac{x}{2}+\frac{1}{12}\sin 6x}.
\end{aligned}$$

If $y=-3$ when $x=0$,

$$\begin{aligned}-3+4 &= Ce^{\frac{0}{2}+\frac{1}{12}\sin 0}\\ \Rightarrow 1 &= C\\ \Rightarrow y+4 &= e^{\frac{x}{2}+\frac{1}{12}\sin 6x}.\end{aligned}$$

When $x=\frac{\pi}{6}$,

$$\begin{aligned}y+4 &= e^{\frac{\pi}{12}+\frac{1}{12}\sin\pi}\\ \Rightarrow y &= e^{\frac{\pi}{12}}-4.\end{aligned}$$

(*b*) (*i*)

$$\begin{aligned}a &= -\frac{v}{100}\\ \Rightarrow \frac{dv}{dt} &= -\frac{v}{100}\\ \Rightarrow \int\frac{1}{v}\,dv &= \int -\frac{1}{100}\,dt\\ \Rightarrow \ln|v| &= -\frac{t}{100}+C\\ \Rightarrow |v| &= e^{-\frac{t}{100}+C}\\ \Rightarrow v &= \pm e^{C}e^{-\frac{t}{100}}\\ \Rightarrow v &= Ce^{-\frac{t}{100}}.\end{aligned}$$

If $v(0)=80$,

$$\begin{aligned}80 &= Ce^{-0}\\ \Rightarrow 80 &= C\\ \Rightarrow v &= 80e^{-\frac{t}{100}}.\end{aligned}$$

(*ii*) If s is the distance travelled by time t,

$$\begin{aligned}v &= 80e^{-\frac{t}{100}}\\ \Rightarrow \frac{ds}{dt} &= 80e^{-\frac{t}{100}}\\ \Rightarrow s &= \int_0^t 80e^{-\frac{t}{100}}\,dt\\ &= -\left.\frac{80e^{-\frac{t}{100}}}{\frac{1}{100}}\right|_0^t\\ &= -8000e^{-\frac{t}{100}}+8000.\end{aligned}$$

(*iii*) To get v in terms of s it's best to go back to the original differential equation but let

$a = v\frac{dv}{ds}$. As $v = 80$ when $s = 0$,

$$
\begin{aligned}
a &= -\frac{v}{100} \\
\Rightarrow v\frac{dv}{ds} &= -\frac{v}{100} \\
\Rightarrow \int_{80}^{v} dv &= \int_{0}^{s} -\frac{1}{100}\, ds \\
\Rightarrow v\Big|_{80}^{v} &= -\frac{s}{100}\Big|_{0}^{s} \\
\Rightarrow v - 80 &= -\frac{s}{100} \\
\Rightarrow v &= 80 - \frac{s}{100}.
\end{aligned}
$$

Question — 2022 Q10.

10. **(a)** A particle moves in a horizontal line such that its speed v at time t is given by the differential equation

$$\frac{dv}{dt} = 5 - 8e^{-t}.$$

(i) Given that $v = 2$ when $t = 0$, find an expression for v in terms of t.

(ii) Find the minimum value of v.

(iii) Find the distance travelled by the particle before it attains its minimum speed.

(b) The rate of decay at any instant of a radioactive substance is proportional to the amount of the substance remaining at that instant. The initial amount of the radioactive substance is N and the amount remaining after time t (hours) is x.

(i) Prove that $x = Ne^{-kt}$, where k is a constant.

(ii) If the initial amount N was reduced to $\frac{N}{3}$ in 14 hours, find the value of k.

(iii) If the amount remaining is reduced from $\frac{N}{3}$ to $\frac{N}{4}$ in t hours, find the value of t.

(a) (i)

$$\begin{aligned}
\frac{dv}{dt} &= 5 - 8e^{-t} \\
\Rightarrow v &= \int 5 - 8e^{-t}\, dt \\
&= 5t + 8e^{-t} + C. \\
v(0) &= 2 \\
\Rightarrow 2 &= 5(0) + 8e^{-0} + C \\
\Rightarrow -6 &= C \\
\Rightarrow v &= 5t + 8e^{-t} - 6.
\end{aligned}$$

(ii) v is minimised when $\frac{dv}{dt} = 0$ (see that $\frac{d^2}{dt^2} = 8e^{-t} > 0$ so any extrema we find will be a minimum).

$$
\begin{aligned}
0 &= 5 - 8e^{-t} \\
\Rightarrow 8e^{-t} &= 5 \\
\Rightarrow e^{-t} &= \frac{5}{8} \\
\Rightarrow -t &= \ln\frac{5}{8} \\
\Rightarrow t &= -\ln\frac{5}{8} \\
&= \ln\frac{8}{5}. \\
v\left(\ln\frac{8}{5}\right) &= 5\ln\frac{8}{5} + 8e^{-\ln\frac{8}{5}} - 6 \\
&= 1.35 \text{ m/s.}
\end{aligned}
$$

(iii)

$$
\begin{aligned}
s &= \int_0^{\ln\frac{8}{5}} v\, dt \\
&= \int_0^{\ln\frac{8}{5}} 5t + 8e^{-t} - 6\, dt \\
&= \frac{5t^2}{2} - 8e^{-t} - 6t \Big|_0^{\ln\frac{8}{5}} \\
&= 0.73 \text{ m.}
\end{aligned}
$$

(b) (i) The question says

$$\frac{dx}{dt} = -k$$

for some constant $k > 0$.

$$
\begin{aligned}
\frac{dx}{dt} &= -kx \\
\Rightarrow \frac{1}{x}dx &= -k\, dt \\
\Rightarrow \int \frac{1}{x}dx &= \int -k\, dt \\
\Rightarrow \ln|x| &= -kt + C \\
\Rightarrow |x| &= e^{-kt+C} \\
&= e^C e^{-kt} \\
&= Ce^{-kt} \\
\Rightarrow x &= \pm Ce^{-kt} \\
&= Ce^{-kt}.
\end{aligned}
$$

$x(0) = N$ so that

$$\begin{aligned} N &= Ce^{-k(0)} \\ \Rightarrow N &= C \\ \Rightarrow x &= Ne^{-kt}. \end{aligned}$$

(ii)

$$\begin{aligned} \frac{N}{3} &= Ne^{-14k} \\ \Rightarrow \frac{1}{3} &= e^{-14k} \\ \Rightarrow \ln\frac{1}{3} &= -14k \\ \Rightarrow \frac{\ln\frac{1}{3}}{-14} &= k \\ \Rightarrow \frac{\ln 3}{14} &= k. \end{aligned}$$

(iii) First we solve $x(s) = \frac{N}{4}$.

$$\begin{aligned} \frac{N}{4} &= Ne^{-ks} \\ \Rightarrow \frac{1}{4} &= e^{-ks} \\ \Rightarrow \ln\frac{1}{4} &= -ks \\ \Rightarrow \frac{\ln 4}{k} &= s \\ \Rightarrow 17.66 &= s. \end{aligned}$$

Therefore

$$\begin{aligned} t &= s - 14 \\ &= 3.66 \text{ hours.} \end{aligned}$$

Question — 2021 Q10.

10. (a) A car of mass 1200 kg starts from rest and travels along a straight horizontal road. The engine of the car exerts a constant power of 3000 W.

If there is no resistance to the motion of the car, find

(i) the speed of the car after 3 minutes

(ii) the average speed of the car during this time.

(b) P, the population of insects in a region, grows at a rate that is proportional to the current population.

$$\frac{dP}{dt} = kP$$

where k is a positive constant. In the absence of any outside factors the population will triple in 15 days.

(i) Find the value of k.

A scientist begins to remove 10 insects from the population each day.

(ii) If there are initially 120 insects in the region the population will not survive. After how many days will the population die out?

(a) (i) As $P = Fv$ where F is the only force acting on the car,

$$\begin{aligned}
3000 &= Fv \\
\Rightarrow \frac{3000}{v} &= F \\
&= ma \\
&= 1200a \\
\Rightarrow \frac{2.5}{v} &= \frac{dv}{dt} \\
\Rightarrow 2.5\, dt &= v\, dv \\
\Rightarrow \int_0^{180} 2.5\, dt &= \int_0^v v\, dv \\
\Rightarrow \Big|_0^{180} 2.5t &= \Big|_0^v \frac{v^2}{2} \\
\Rightarrow 450 &= \frac{v^2}{2} \\
\Rightarrow 900 &= v^2 \\
\Rightarrow 30 &= v.
\end{aligned}$$

(ii) The average speed is the distance travelled divided by time. We therefore want s when $v = 30$ or $t = 180$ (whichever is easier). Looking at our work in part (i) and ignoring the definite part of the integral,

$$\begin{aligned}2.5t + C &= \frac{v^2}{2}\\ \Rightarrow 5t + C &= v^2.\end{aligned}$$

As $v = 0$ when $t = 0$,

$$\begin{aligned}5(0) + C &= 0^2\\ \Rightarrow C &= 0\\ \Rightarrow 5t &= v^2\\ \Rightarrow \sqrt{5t} &= v\\ \Rightarrow \sqrt{5}\sqrt{t} &= \frac{ds}{dt}\\ \Rightarrow \sqrt{5}t^{\frac{1}{2}} &= \frac{ds}{dt}\\ \Rightarrow \sqrt{5}\frac{t^{\frac{3}{2}}}{\frac{3}{2}}\Bigg|_0^{180} &= s\\ \Rightarrow \frac{2\sqrt{5}}{3}(180)^{\frac{3}{2}} &= s\\ \Rightarrow 3600 &= s.\end{aligned}$$

The average speed is then

$$\begin{aligned}\frac{s}{t} &= \frac{3600}{180}\\ &= 20 \text{ m/s}.\end{aligned}$$

Note: We could have also used $\frac{3000}{v} = 1200a$ by letting $a = v\frac{dv}{ds}$ to find s when $v = 30$.

(b) (i)

$$\begin{aligned}\frac{dP}{dt} &= kP\\ \Rightarrow \frac{1}{P}dP &= k\,dt\\ \Rightarrow \int \frac{1}{P}dP &= \int k\,dt\\ \Rightarrow \ln|P| &= kt + C\\ \Rightarrow |P| &= e^{kt+C}\\ &= e^C e^{kt}\\ &= Ce^{kt}\\ \Rightarrow P &= \pm Ce^{kt}\\ &= Ce^{kt}.\end{aligned}$$

If $P(0) = N$, $P(15) = 3N$ so that

$$\begin{aligned} N &= Ce^{k(0)} \\ \Rightarrow N &= C \\ \Rightarrow P &= Ne^{kt} \\ \Rightarrow 3N &= Ne^{15k} \\ \Rightarrow 3 &= e^{15k} \\ \Rightarrow \ln 3 &= 15k \\ \Rightarrow \frac{\ln 3}{15} &= k. \end{aligned}$$

(ii) In this case

$$\begin{aligned} \frac{dP}{dt} &= kP - 10 \\ \Rightarrow \frac{dP}{kP-10} &= dt. \end{aligned}$$

We have $P(0) = 120$ and want to know t when $P(t) = 0$.

$$\begin{aligned} \frac{dP}{kP-10} &= dt \\ \Rightarrow \int_{120}^{0} \frac{dP}{kP-10} &= \int_{0}^{t} dt \\ \Rightarrow \frac{1}{k} \ln|kP-10| \Big|_{120}^{0} &= t \Big|_{0}^{t} \\ \Rightarrow \frac{1}{k} \ln|-10| - \frac{1}{k} \ln|120k-10| &= t \\ \Rightarrow 28.82 \text{ days} &= t. \end{aligned}$$

(iii)

$$\begin{aligned} \frac{N}{3} &= Ne^{-14k} \\ \Rightarrow \frac{1}{3} &= e^{-14k} \\ \Rightarrow \ln\frac{1}{3} &= -14k \\ \Rightarrow \frac{\ln\frac{1}{3}}{-14} &= k \\ \Rightarrow \frac{\ln 3}{14} &= k. \end{aligned}$$

Question — 2020 Q10.

10. (a) One method of dyeing a piece of cloth is to immerse it in a container which has P grams of dye dissolved in a fixed volume of water.

The cloth absorbs the dye at a rate proportional to the mass of dye remaining.

$$\frac{dx}{dt} = k(P - x)$$

where t is time in seconds, x is the mass of dye absorbed by the cloth and $k = \frac{1}{50}$.

(i) Find the time taken to dye a piece of cloth if a mass of $\frac{5}{8}P$ needs to be absorbed to reach the desired colour.
(Note: $\int \frac{dx}{a+bx} = \frac{1}{b}\ln|a + bx| + c$)

An alternative method is to keep the mass of dye present in the water constant at P grams by continuously adding dye throughout the process.

(ii) Find the time taken to dye the piece of cloth to the desired colour using this method.

(b) A particle P travelling in a straight line has a deceleration of $4v^{n+1}$ m s^{-2}, where n (> 0) is a constant and v is its speed at time t (> 0).

P has an initial speed of u.

(i) Find an expression for v in terms of u, n and t.

(ii) When $n = 3$ obtain an expression for the speed of P when it has travelled a distance of 3 m from its initial position.

(a) (i) Assuming $x(0) = 0$ (i.e. there is no dye at the beginning of the immersion),

$$\frac{dx}{dt} = \frac{1}{50}(P - x)$$
$$\Rightarrow \frac{dx}{P-x} = \frac{1}{50}\,dt$$
$$\Rightarrow \int_0^{\frac{5P}{8}} \frac{dx}{P-x} = \int_0^t \frac{1}{50}\,dt$$
$$\Rightarrow -\ln|P-x|\Big|_0^{\frac{5P}{8}} = \frac{t}{50}\Big|_0^t$$
$$\Rightarrow -\ln\left|\frac{3P}{8}\right| + \ln|P| = \frac{t}{50}$$
$$\Rightarrow \ln\left|\frac{P}{\frac{3P}{8}}\right| = \frac{t}{50}$$
$$\Rightarrow 50\ln\frac{8}{3}\text{ sec} = t.$$

(ii) In this case

$$\frac{dx}{dt} = \frac{1}{50}P$$
$$\Rightarrow x = \frac{1}{50}Pt.$$

If $x(t) = \frac{5P}{8}$,

$$\frac{5P}{8} = \frac{1}{50}Pt$$
$$\Rightarrow \frac{125}{4}\ \text{sec} = t.$$

(b) (i)

$$a = -4v^{n+1}$$
$$\Rightarrow \frac{dv}{dt} = -4v^{n+1}$$
$$\Rightarrow \frac{1}{v^{n+1}}\ dv = -4\ dt$$
$$\Rightarrow \int_u^v v^{-n-1}\ dv = \int_0^t -4\ dt$$
$$\Rightarrow \left.\frac{v^{-n}}{-n}\right|_u^v = -4t\Big|_0^t$$
$$\Rightarrow -\frac{1}{nv^n} + \frac{1}{nu^n} = -4t$$
$$\Rightarrow \frac{1}{nu^n} + 4t = \frac{1}{nv^n}$$
$$\Rightarrow \frac{1}{nu^n} + \frac{4tnu^n}{nu^n} = \frac{1}{nv^n}$$
$$\Rightarrow \frac{1+4tnu^n}{nu^n} = \frac{1}{nv^n}$$
$$\Rightarrow \frac{nu^n}{1+4tnu^n} = nv^n$$
$$\Rightarrow \frac{u^n}{1+4tnu^n} = v^n$$
$$\Rightarrow \frac{u}{\sqrt[n]{1+4tnu^n}} = v.$$

(ii) Now

$$\frac{u}{\sqrt[3]{1+12tu^3}} = v$$

and we want v when $s = 3$. We could integrate $v(t)$ to find $s(t)$, find t when $s = 3$ and go back to finding v. But any differential equation involving a can be written as a differential equation in v and s. This is better, as we want v for a certain value of s and don't actually care about time. Considering v as a function of s, with $s = 0$ at the beginning of the movement, $v(0) = u$ and

$$
\begin{aligned}
a &= -4v^4 \\
\Rightarrow v\frac{dv}{ds} &= -4v^4 \\
\Rightarrow \frac{1}{v^3}\,dv &= -4\,ds \\
\Rightarrow \int_u^v v^{-3}\,dv &= \int_0^3 -4\,ds \\
\Rightarrow \left.\frac{v^{-2}}{-2}\right|_u^v &= -4s\Big|_0^3 \\
\Rightarrow -\frac{1}{2v^2} + \frac{1}{2u^2} &= -12 \\
\Rightarrow \frac{1}{2u^2} + \frac{24u^2}{2u^2} &= \frac{1}{2v^2} \\
\Rightarrow \frac{1+24u^2}{2u^2} &= \frac{1}{2v^2} \\
\Rightarrow \frac{2u^2}{1+24u^2} &= 2v^2 \\
\Rightarrow \frac{u^2}{1+24u^2} &= v^2 \\
\Rightarrow \frac{u}{\sqrt{1+24u^2}} &= v.
\end{aligned}
$$

Question — 2019 Q10.

10. **(a)** A particle P moves along a straight line.
The speed of P at time t is v, where $v = at^2 + bt + c$ and a, b and c are constants.
The initial speed of the particle is 15 m s^{-1}.
After 2·5 seconds the particle reaches its **minimum** speed of 2·5 m s^{-1}.

Find
(i) the value of a, the value of b, and the value of c

(ii) the acceleration of P when $t = 4$ seconds

(iii) the distance travelled by P in the third second of the motion.

(b) A particle, of mass m falls vertically downwards under gravity.
At time t, the particle has speed v and it experiences a resistance force of magnitude kmv, where k is a constant.
The initial speed of the particle is u.

(i) Show that $v = \frac{g}{k} - \left(\frac{g}{k} - u\right)e^{-kt}$, at time t.

(ii) If $u = 9{\cdot}8$ m s^{-1} and $k = 0{\cdot}98$ s^{-1}, find the distance travelled by the particle in 4 seconds.

(Note: $\int \frac{dx}{a+bx} = \frac{1}{b}\ln|a+bx| + c$).

(a) (i) We are told three things: $v(0) = 15$, $v(2.5) = 2.5$ and $v'(2.5) = 0$. This will give us three equations in three variables a, b and c.

$$\begin{aligned}
v(0) &= 15\\
\Rightarrow 15 &= a(0)^2 + b(0) + c\\
\Rightarrow 15 &= c.\\
v(2.5) &= 2.5\\
\Rightarrow 2.5 &= a(2.5)^2 + b(2.5) + 15\\
\Rightarrow -12.5 &= 6.25a + 2.5b\\
\Rightarrow -50 &= 25a + 10b\\
\Rightarrow -10 &= 5a + 2b.\\
v'(t) &= 2at + b.\\
v'(2.5) &= 0\\
\Rightarrow 0 &= 2a(2.5) + b\\
\Rightarrow -5a &= b.
\end{aligned}$$

Substituting $b = -5a$ into $-10 = 5a + 2b$ gives

$$\begin{aligned} -10 &= 5a + 2(-5a) \\ \Rightarrow -10 &= -5a \\ \Rightarrow 2 &= a \\ \Rightarrow b &= -5(2) \\ &= -10. \end{aligned}$$

(ii)

$$\begin{aligned} v(t) &= 2t^2 - 10t + 15 \\ \Rightarrow a(t) &= v'(t) \\ &= 4t - 10 \\ \Rightarrow a(4) &= 6. \end{aligned}$$

(iii) The distance travelled in the third second of motion is equal to

$$\begin{aligned} \int_2^3 v(t)\,dt &= \int_2^3 2t^2 - 10t + 15\,dt \\ &= \left(\frac{2t^3}{3} - 5t^2 + 15t\right)\Bigg|_2^3 \\ &= \left(\frac{2(3)^3}{3} - 5(3)^2 + 15(3)\right) - \left(\frac{2(2)^3}{3} - 5(2)^2 + 15(2)\right) \\ &= 18 - \frac{46}{3} \\ &= \frac{8}{3}. \end{aligned}$$

(b) (i) Taking the downwards direction as positive, the resistance force acts upwards and so

$$\begin{aligned} F &= ma \\ \Rightarrow mg - kmv &= ma \\ \Rightarrow g - kv &= \frac{dv}{dt} \\ \Rightarrow \int dt &= \int \frac{1}{g - kv}\,dv \\ t + C &= \frac{1}{-k}\ln|g - kv| \\ \Rightarrow -kt + C &= \ln|g - kv| \\ \Rightarrow e^{-kt+C} &= |g - kv| \\ \Rightarrow \pm e^C e^{-kt} &= g - kv \\ \Rightarrow kv &= g + Ce^{-kt} \\ \Rightarrow v &= \frac{g}{k} + Ce^{-kt}. \end{aligned}$$

If $v(0) = u$, then

$$\begin{aligned} u &= \frac{g}{k} + Ce^{-k(0)} \\ \Rightarrow u - \frac{g}{k} &= C \\ \Rightarrow v &= \frac{g}{k} + \left(u - \frac{g}{k}\right) e^{-kt} \\ &= \frac{g}{k} - \left(\frac{g}{k} - u\right) e^{-kt}. \end{aligned}$$

(ii) In this case

$$\begin{aligned} v(t) &= \frac{9.8}{0.98} - \left(\frac{9.8}{0.98} - 9.8\right) e^{-0.98t} \\ &= 10 - 0.2e^{-0.98t} \\ \Rightarrow \text{Distance travelled} &= \int_0^4 10 - 0.2e^{-0.98t}\, dt \\ &= \left(10t + \frac{0.2}{0.98}e^{-0.98t}\right)\Big|_0^4 \\ &= \left(10(4) + \frac{10}{49}e^{-0.98(4)}\right) - \left(0 + \frac{10}{49}\right) \\ &= 39.8 \text{ m/s}. \end{aligned}$$

Question — 2018 Q10.

10. (a) If $\frac{dy}{dx} = 3\sin 3x + \cos 5x$ and $y = 1$ when $x = \frac{\pi}{4}$, find the value of y when $x = \frac{\pi}{2}$.
Give your answer correct to 2 decimal places.

(b) If there were no emigration, the population x of a certain county would increase at a constant rate of 2·5% per annum. By emigration the county loses population at a constant rate of n people per annum.
When the time is measured in years then $\frac{dx}{dt} = \frac{x}{40} - n$.

(i) If initially the population is P people, find in terms of n, P and t, the population after t years.

(ii) Given that $n = 800$ and $P = 30000$, find the value of t when the population is 29 734.

(a)

$$\begin{aligned}
\frac{dy}{dx} &= 3\sin 3x + \cos 5x \\
\Rightarrow \int_1^y dy &= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} 3\sin 3x + \cos 5x\, dx \\
\Rightarrow y\Big|_1^y &= \left(-\cos 3x + \frac{1}{5}\sin 5x\right)\Big|_{\frac{\pi}{4}}^{\frac{\pi}{2}} \\
\Rightarrow y - 1 &= \left(-\cos 3\left(\frac{\pi}{2}\right) + \frac{1}{5}\sin 5\left(\frac{\pi}{2}\right)\right) - \left(-\cos 3\left(\frac{\pi}{4}\right) + \frac{1}{5}\sin 5\left(\frac{\pi}{4}\right)\right) \\
\Rightarrow y &= 1 + \frac{1}{5} - \left(\frac{1}{\sqrt{2}} - \frac{1}{5\sqrt{2}}\right) \\
&= \frac{6}{5} - \frac{4}{5\sqrt{2}}.
\end{aligned}$$

(b) (i)

$$\frac{dx}{dt} = \frac{x}{40} - n$$
$$\Rightarrow \int \frac{1}{\frac{x}{40} - n}\, dx = \int dt$$
$$\Rightarrow \frac{1}{\frac{1}{40}} \ln\left|\frac{x}{40} - n\right| = t + C$$
$$\Rightarrow \ln\left|\frac{x}{40} - n\right| = \frac{t}{40} + C$$
$$\Rightarrow \left|\frac{x}{40} - n\right| = e^{\frac{t}{40} + C}$$
$$\Rightarrow \frac{x}{40} - n = \pm e^{C} e^{\frac{t}{40}}$$
$$\Rightarrow \frac{x}{40} = n + Ce^{\frac{t}{40}}$$
$$\Rightarrow x = 40n + Ce^{\frac{t}{40}}.$$

If $x(0) = P$,

$$P = 40n + Ce^{\frac{0}{40}}$$
$$\Rightarrow P - 40n = C$$
$$\Rightarrow x = 40n + (P - 40n)e^{\frac{t}{40}}.$$

(ii)

$$29734 = 40(800) + (30000 - 40(800))e^{\frac{t}{40}}$$
$$\Rightarrow 29734 = 32000 - 2000e^{\frac{t}{40}}$$
$$\Rightarrow -2268 = -2000e^{\frac{t}{40}}$$
$$\Rightarrow \frac{567}{500} = e^{\frac{t}{40}}$$
$$\Rightarrow \ln\frac{567}{500} = \frac{t}{40}$$
$$\Rightarrow 40\ln\frac{567}{500} = t.$$

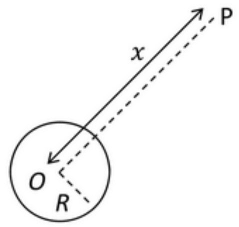
Question — 2017 Q10.

10. (a) A particle starts from rest and moves in a straight line with acceleration $(25 - 10v)$ m s^{-2}, where v is the speed of the particle.

(i) After time t, find v in terms of t. (Note: $\int \frac{dx}{a+bx} = \frac{1}{b} ln|a + bx| + c$).

(ii) Find the time taken to acquire a speed of 2·25 m s^{-1} and find the distance travelled in this time.

(b) A spacecraft P of mass m moves in a straight line towards O, the centre of the earth.
The radius of the earth is R.
When P is a distance x from O, the force exerted by the earth on P is directed towards O and has magnitude $\frac{k}{x^2}$, where k is a constant.



(i) Show that $k = mgR^2$.

P starts from rest when its distance from O is $5R$.

(ii) Find, in terms of R, the speed of P as it hits the surface of the earth, given that air resistance can be ignored.

(a) (i)

$$
\begin{aligned}
a &= 25 - 10v \\
\Rightarrow \frac{dv}{dt} &= 25 - 10v \\
\Rightarrow \int \frac{1}{25-10v}\,dv &= \int dt \\
\Rightarrow \frac{1}{-10}\ln|25-10v| &= t + C \\
\Rightarrow \ln|25-10v| &= -10t + C \\
\Rightarrow |25-10v| &= e^{-10t+C} \\
\Rightarrow 25 - 10v &= \pm e^C e^{10t} \\
\Rightarrow -10v &= -25 + Ce^{-10t} \\
\Rightarrow v &= 2.5 + Ce^{-10t}.
\end{aligned}
$$

If the particle starts from rest $v(0) = 0$ and so

$$
\begin{aligned}
0 &= 2.5 + Ce^{-10(0)} \\
\Rightarrow -2.5 &= C \\
\Rightarrow v &= 2.5 - 2.5e^{-10t}.
\end{aligned}
$$

(ii)

$$
\begin{aligned}
v &= 2.25 \\
\Rightarrow 2.5 - 2.5e^{-10t} &= 2.25 \\
\Rightarrow 0.25 &= 2.5e^{-10t} \\
\Rightarrow 0.1 &= e^{-10t} \\
\Rightarrow \ln 0.1 &= -10t \\
\Rightarrow \frac{\ln 0.1}{-10} &= t \\
\Rightarrow 0.23 \text{ seconds} &= t.
\end{aligned}
$$

$$
\begin{aligned}
v &= 2.5 - 2.5e^{-10t} \\
\Rightarrow \frac{ds}{dt} &= 2.5 - 2.5e^{-10t} \\
\Rightarrow \text{Distance travelled} &= \int_0^{0.23} 2.5 - 2.5e^{-10t}\, dt \\
&= 2.5t + 0.25e^{-10t} \Big|_0^{0.23} \\
&= 0.35 \text{ m}.
\end{aligned}
$$

(b) (i) We know that when $x = R$, i.e. when the spacecraft is on the surface of the earth, that the gravitational force is equal to mg. Therefore

$$
\begin{aligned}
mg &= \frac{k}{R^2} \\
\Rightarrow mgR^2 &= k.
\end{aligned}
$$

(ii) We have that $x = 5R$ and speed $v = 0$ when $t = 0$, and that the gravitational force acts towards the centre of the earth. Taking the direction outwards from earth to be positive, this force acts in the negative direction and so

$$
\begin{aligned}
F &= -\frac{k}{x^2} \\
\Rightarrow ma &= -\frac{mgR^2}{x^2} \\
\Rightarrow v\frac{dv}{dx} &= -\frac{gR^2}{x^2} \\
\Rightarrow \int v\, dv &= \int -gR^2x^{-2}\, dx \\
\Rightarrow \frac{v^2}{2} &= gR^2x^{-1} + C.
\end{aligned}
$$

The initial conditions $v = 0$ when $x = 5R$ give

$$\begin{aligned} 0 &= \frac{gR^2}{5R} + C \\ \Rightarrow -\frac{gR}{5} &= C \\ \Rightarrow \frac{v^2}{2} &= \frac{gR^2}{x} - \frac{gR}{5}. \end{aligned}$$

We want v when $x = R$.

$$\begin{aligned} \frac{v^2}{2} &= \frac{gR^2}{R} - \frac{gR}{5} \\ &= \frac{4gR}{5} \\ \Rightarrow v^2 &= \frac{8gR}{5} \\ \Rightarrow v &= \sqrt{\frac{8gR}{5}}. \end{aligned}$$

Question — 2016 Q10.

10. **(a)** At time t seconds the acceleration a m s^{-2} of a particle, P, is given by

$$a = 8t + 4.$$

At $t = 0$, P passes through a fixed point with velocity -24 m s^{-1}.

(i) Show that P changes its direction of motion only once in the subsequent motion.

(ii) Find the distance travelled by P between $t = 0$ and $t = 3$.

(b) A particle moves along a straight line in such a way that its acceleration is always directed towards a fixed point O on the line, and is proportional to its displacement from that point.
The displacement of the particle from O at time t is x.
The equation of motion is

$$v\frac{dv}{dx} = -\omega^2 x$$

where v is the velocity of the particle at time t and ω is a constant.

The particle starts from rest at a point P, a distance A from O.

Derive an expression for
(i) v in terms of A, ω and x
(ii) x in terms of A, ω and t.

(a) (i) P changes direction when v changes sign, i.e. when $v = 0$ but $a \neq 0$.

$$\begin{aligned} a &= 8t + 4 \\ \Rightarrow \frac{dv}{dt} &= 8t + 4 \\ \Rightarrow \int_{-24}^{v} dv &= \int_0^t 8t + 4\, dt \\ \Rightarrow v\Big|_{-24}^{v} &= 4t^2 + 4t\Big|_0^t \\ \Rightarrow v + 24 &= 4t^2 + 4t \\ \Rightarrow v &= 4t^2 + 4t - 24. \end{aligned}$$

If $v = 0$,

$$\begin{aligned} 4t^2 + 4t - 24 &= 0 \\ \Rightarrow t^2 + t - 6 &= 0 \\ \Rightarrow (t-2)(t+3) &= 0 \\ \Rightarrow t &= 2, \cancel{-3}. \end{aligned}$$

At $t = 2$ $a = 12 \neq 0$ and so the object changes direction at this time.

(ii) The object changes direction at $t = 2$, so if we want the **distance travelled** and not the displacement, we must separately calculate the distance travelled between times 0 and 2, and 2 and 3.

$$\begin{aligned}
v &= 4t^2 + 4t - 24 \\
\Rightarrow \frac{ds}{dt} &= 4t^2 + 4t - 24 \\
\Rightarrow \text{Displacement between } t = 0 \text{ and } t = 2 &= \int_0^2 4t^2 + 4t - 24\, dt \\
&= \left(\frac{4t^3}{3} + 2t^2 - 24t \right) \Big|_0^2 \\
&= \frac{-88}{3}. \\
\text{Displacement between } t = 2 \text{ and } t = 3 &= \int_2^3 4t^2 + 4t - 24\, dt \\
&= \left(\frac{4t^3}{3} + 2t^2 - 24t \right) \Big|_2^3 \\
&= -18 - \frac{-88}{3}. \\
&= \frac{34}{3}.
\end{aligned}$$

Therefore the total distance travelled is

$$\left| \frac{-88}{3} \right| + \frac{34}{3} = \frac{122}{3} \text{ m.}$$

Question — 2015 Q10.

10. **(a)** Two cars, A and B, start from rest at O and begin to travel in the same direction.

The speeds of the cars are given by $v_A = t^2$ and $v_B = 6t - 0.5t^2$, where v_A and v_B are measured in m s^{-1} and t is the time in seconds measured from the instant when the cars started moving.

(i) Find the speed of each car after 4 seconds.

(ii) Find the distance between the cars after 4 seconds.

(iii) On the same speed-time graph, sketch the speed of A and the speed of B for the first 4 seconds and shade in the area that represents the distance between the cars after 4 seconds.

(b) A company uses a cost function $C(x)$ to estimate the cost of producing x items. The cost function is given by the equation $C(x) = F + V(x)$ where F is the estimate of all fixed costs and $V(x)$ is the estimate of the variable costs (energy, materials, etc.) of producing x items.

$\frac{dC}{dx} = M(x)$ is the marginal cost, the cost of producing one more item.

A certain company has a marginal cost function given by $M(x) = 74 + 1.1x + 0.03x^2$.

(i) Find the cost function, $C(x)$.
(ii) Find the increase in cost if the company decides to produce 160 items instead of 120.
(iii) If $C(10) = 3500$, find the fixed costs.

(a) (i)

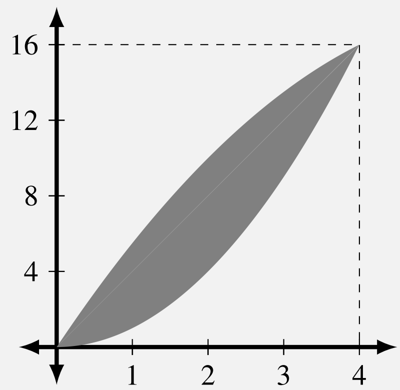
$$\begin{aligned} v_A(4) &= 4^2 \\ &= 16, \\ v + B &= 6(4) - 0.5(4)^2 \\ &= 16. \end{aligned}$$

(ii) If s_A, s_B are the displacement of A, B from O, then $s_A(0) = s_B(0) = 0$ and

$$\begin{aligned} v_A &= t^2 \\ \Rightarrow s_A(4) &= \int_0^4 t^2\, dt \\ &= \left.\frac{t^3}{3}\right|_0^4 \\ &= \frac{64}{3}. \\ v_B &= 6t - 0.5t^2 \\ \Rightarrow s_B(4) &= \int_0^4 6t - 0.5t^2\, dt \\ &= \left.\left(3t^2 - \frac{1}{6}t^3\right)\right|_0^4 \\ &= \frac{112}{3}. \end{aligned}$$

Therefore the distance between them is $\frac{112}{3} - \frac{64}{3} = 16$ m.

(iii)



(b) (i)

$$\begin{aligned} M(x) &= 74 + 1.1x + 0.03x^2 \\ \Rightarrow \frac{dC}{dx} &= 74 + 1.1x + 0.03x^2 \\ \Rightarrow C(x) &= \int 74 + 1.1x + 0.03x^2\, dx \\ \Rightarrow C(x) &= 74x + 0.55x^2 + 0.01x^3 + D. \end{aligned}$$

Under the assumption that $V(0) = 0$ and so $C(0) = F$,

$$\begin{aligned} F &= D \\ \Rightarrow C(x) &= 74x + 0.55x^2 + 0.01x^3 + F. \end{aligned}$$

(ii)

$$
\begin{aligned}
& C(160)-C(120) \\
=& \left(74(160)+0.55(160)^2+0.01(160)^3+F\right) \\
& -\left(74(120)+0.55(120)^2+0.01(120)^3+F\right) \\
=& 32800.
\end{aligned}
$$

(iii)

$$
\begin{aligned}
& C(10)=3500 \\
\Rightarrow\ & 3500=74(10)+0.55(10)^2+0.01(10)^3+F \\
\Rightarrow\ & 2695=F.
\end{aligned}
$$