Difference Equations Exam Questions

Note These exam questions are given in reverse chronological order as they appear in exam papers; 2023 (deferred), 2023, Sample paper. Because Difference Equations were not on the old syllabus there are relatively few exam papers to practice with. As a result, after the sample paper questions there are 10 more questions written by me. They are designed not only to be of the same difficuly as Leaving Cert questions, but also written in the same style. For each of these questions, the number of marks they would be graded out of is given so that students have an idea of how long the question should take them (remember that students have 150/8=18.75 minutes to do a full question and approximately 8-10 minutes to do a part (a) or (b) question). A difficulty grade out of 10 is also given so that students have an idea of how challenging the question is relatively to the other Leaving Cert questions.

Question — 2023 Deferred Q2(b).

(b) In another, larger computer network, a message travels from one computer to another in the network. Each time a message travels from one computer to the next the number of errors in the message, E, increases by 15%. However C errors are corrected each time the message travels. The number of computers the message travels to is counted using the number n.

A message starts at computer $n = 0$ and travels on a linear path through the computer network.

E, the number of errors in the message, may be modelled by the difference equation:

$$E_{n+1} = 1.15E_n - C$$

where $n \geq 0, n \in \mathbb{Z}$.

There are 101 errors in the message when it leaves computer 0, i.e. $E_0 = 101$.

(i) Solve this difference equation to find an expression for E_n in terms of n and C.

(ii) It is found that the message contains zero errors after it reaches the 21st computer, i.e. $E_{21} = 0$. Calculate the value of C to the nearest whole number.

(iii) E may also be modelled using a differential equation. Write a differential equation for $\frac{dE}{dn}$, the rate of change of E with respect to n, in terms of E and C.

Question — 2023 Deferred Q3.

Question 3

In an economic model, the gross national income G of a country, consists of three separate contributions:

$$G = P + I + S$$

where P represents private spending by citizens, I represents investment in the economy, and S represents government spending.

G can be modelled using a difference equation, where P and I change each year n and where S is assumed to be constant. That is:

$$G_n = P_n + I_n + S$$

In any year, P is proportional to the value of G for the previous year. That is:

$$P_{n+1} = aG_n$$

where $n \geq 0, n \in \mathbb{Z}$ and $a = \frac{15}{16}$.

In any year, I is proportional to the change in the value of P between that year and the previous one. That is:

$$I_{n+1} = b(P_{n+1} - P_n)$$

where $n \geq 0, n \in \mathbb{Z}$ and $b = \frac{3}{5}$.

(i) Use this information to form a second-order inhomogeneous difference equation for G and express it in the form:

$$G_{n+2} + cG_{n+1} + dG_n = S$$

for the constants c, d which are to be determined.

Calculate the values for c and d.

(ii) Assuming the government spends no money (i.e. assuming $S = 0$ euro), G can be expressed by the second-order homogeneous difference equation:

$$G_{n+2} + cG_{n+1} + dG_n = 0$$

Using $G_0 = 840$ and $G_1 = 820$ in billions of euros, solve this difference equation to find an expression for G_n in terms of n.

Calculate G_6 to the nearest billion euros.

(iii) Assuming the government spends 40 billion euros each year (i.e. assuming $S = 40$ in billions of euros), G can be expressed by the second-order inhomogeneous difference equation:

$$G_{n+2} + cG_{n+1} + dG_n = 40$$

Again using $G_0 = 840$ and $G_1 = 820$ in billions of euros, solve this difference equation to find an expression for G_n in terms of n.

Again calculate G_6 to the nearest billion euros.

Question — 2023 Q6.

Question 6

Spider plants (*Chlorophytum comosum*) can reproduce asexually, producing new plants called 'spiderettes' or 'pups'. The manager of a garden centre is told that a one year old spider plant produces two pups each year, that a two year old spider plant produces three pups each year, and that spider plants which are less than one year old or more than two years old do not produce any pups.

The manager predicts that U, the number of pups produced in the garden centre in any year can be expressed by the second-order homogeneous difference equation:

$$U_{n+2} = 2U_{n+1} + 3U_n$$

where $n \geq 0, n \in \mathbb{Z}, U_0 = 1$ and $U_1 = 2$.

(i) Write down the values of U_2 and U_3.

(ii) Solve the difference equation to find an expression for U_n in terms of n.

(iii) Calculate U_{10}.

The manager realises that this model does not take into account the sale of any of the spider plants produced in the garden centre. The manager decides that the garden centre will not sell any of the spider plants in either of the first two years, but that $2n$ of the new pups will be sold in each year n after that.

As part of an improved model, the manager now predicts that V, the number of pups produced and retained (not sold) in the garden centre in any year can be expressed by the second-order inhomogeneous difference equation:

$$V_{n+2} = 2V_{n+1} + 3V_n - 2(n + 2)$$

where $n \geq 0, n \in \mathbb{Z}, V_0 = 1$ and $V_1 = 2$.

(iv) Solve this new difference equation to find an expression for V_n in terms of n.

(v) Calculate V_{10}.

Question — 2023 Q10(a).

Question 10

(a) An entomologist (a scientist who studies insects) maintains a population of grasshoppers in her laboratory.

The entomologist's research tells her that the population of this species of grasshopper should increase by a factor of 1.2 each month if they are left undisturbed. However the entomologist removes 30 grasshoppers from the population each month, to carry out research on them.

The entomologist develops a difference equation model to predict U_n, the number of grasshoppers present at the beginning of month n.

At the start of the first month the entomologist has 175 grasshoppers, i.e. $U_0 = 175$.

(i) Calculate the values of U_1 and U_2.

(ii) Write down a difference equation to express U_{n+1} in terms of U_n, where $n \geq 0, n \in \mathbb{Z}$.

(iii) Solve this difference equation to find an expression for U_n in terms of n.

(iv) Calculate U_{12}, the number of grasshoppers which the model predicts will be in the population after one year.

Question — Sample Paper Q1(b).

(b) A gardener plants a new fruit tree which has three new branches. In a branch's first year of growth it will not produce any additional branches. Each branch will produce one additional branch every year after that.

The gardener models this growth pattern by defining U_n to be the number of branches on the tree n years after planting, with $U_0 = 3$ and $U_1 = 3$.

(i) Write down the values of U_2 and U_3.

(ii) Write down a difference equation for U_{n+2} in terms of U_{n+1} and U_n, where $n \geq 0, n \in \mathbb{Z}$.

(iii) Solve this difference equation to find an expression for U_n in terms of n.

(iv) Plants must be cut back regularly to allow them room to grow. How many of the old branches should be removed at the end of year 4 to ensure that there are exactly 14 branches at the end of year 5?

Question — Sample Paper Q8.

Question 8

A group of scientists are investigating the population, P, of rabbits on a certain island. They estimate that there are 8000 rabbits on the island and that the population is growing at a constant rate of 3% per year.

The scientists plan to remove a number of rabbits from the island every year, to help populate another habitat. They develop mathematical models to predict how P will change if B rabbits are removed from the island every year.

The first model which the scientists develop uses a difference equation to express the population of rabbits in year $n+1$ in terms of the population in year n.

The difference equation is:

$$P_{n+1} = 1.03P_n - B$$

where $n \geq 0$, $n \in \mathbb{Z}$ and $P_0 = 8000$.

(i) Solve this difference equation to find an expression for P_n in terms of n and B.

The second model which the scientists develop uses a differential equation to express the rate of change of P with respect to n, time measured in years.

The differential equation is:

$$\frac{dP}{dn} = 0.03P - B$$

where $n \geq 0$, $n \in \mathbb{R}$ and $P(0) = 8000$.

(ii) Solve this differential equation to find an expression for P in terms of n and B.

The scientists want to know what each model predicts the rabbit population on the island will be after 50 years, if 200 rabbits are removed each year.

(iii) Calculate P_{50} using the first model and $P(50)$ using the second model, when $B = 200$.

(iv) Each of these models makes a different assumption about the removal of the rabbits from the island. What are the two different assumptions?

(v) The scientists want to know what value of B should be chosen so as to keep the rabbit population on the island constant. Calculate this value of B using either model.

Question — 20 marks, 4/10 difficulty. David takes over management of a forest. He decides to chop down 10% of his trees during each year. However he also plants 50 trees at the end of each year.

David wants to develop a difference equation model to predict T_n, the number of trees in the forest n years after David takes over management of the forest.

At the start of the first year the forest contains 1000 trees, i.e. $T_0 = 1000$.

(i) Write down a difference equation to express T_{n+1} in terms of T_n, where $n \geq 0$, $n \in \mathbb{Z}$.

(ii) Solve this difference equation to find an expression for T_n in terms of n.

(iii) Calculate T_{20}, the number of trees in the forest 20 years after David takes over.

Question — 50 marks, 8/10 difficuly. Emily opens up a savings account. Each year the amount of money in her savings account accrues 4% interest, and she also adds €2000 to the savings account each year. M_n, the amount of money in her savings account n years after she opens it, can be modelled by the difference equation

$$M_{n+1} = 1.04M_n + 2000,$$

where $n \geq 0$, $n \in \mathbb{Z}$.

Emily starts off with nothing in her account, so $M_0 = 0$.

(i) Solve this difference equation to find M_n in terms of n.

(ii) Calculate M_{10}, the amount of money Emily will have in her account after 10 years.

(iii) When Emily has €30,000 in her account she will be able to buy a car. If she is able to afford a car k years after she opens the account, calculate k.

(iv) Emily instead decides to model her savings as a differential equation. Find an expression for $\frac{dM}{dn}$, the rate of change of the amount of money in her account with respect to n (measured in years).

(v) Emily decides this is not a realistic model, as later in life she expects to be able to save more money. In her first year she will add €2000 to the bank account, however she will increase this amount by 5% each year. If L_n is the amount of money she has in her account n years after she opens the account, then L can be modelled with the new difference equation

$$L_{n+1} = 1.04L_n + 2000(1.05)^n.$$

Solve this difference equation to find L_n in terms of n.

(vi) Calculate L_{10}, the amount of money Emily will have in her account after 10 years.

Question — 50 marks, 7/10 difficulty. A difference equation for the sequence U is given by

$$U_{n+2} = U_{n+1}U_n^2, \quad U_0 = 1, U_1 = 2.$$

where $n \geq 0$, $n \in \mathbb{Z}$.

(i) Calculate U_2 and U_3.

(ii) Let $V_n = \log_2 U_n$. Use the difference equation with initial conditions for U

$$U_{n+2} = U_{n+1}U_n^2, \quad U_0 = 1,\ U_1 = 2.$$

to write down a difference equation with initial conditions for V.

(iii) Solve this difference equation to get V_n as a function of n.

(iv) Use your answer to part (iii) to get U_n as a function of n.

(v) Find U_{10}. Give your answer in the form 2^a for some $a \in \mathbb{Z}$.

Question — 30 marks, 5/10 difficulty. Newton's Law of Heating and Cooling states that the change in temperature of an object per unit time is proportional to the difference between the temperature of the object and the temperature of its surroundings.

A hot chicken is removed from the oven and is left to rest in the kitchen, which is at a constant temperature of 20°. Its temperature n minutes after it is removed from the oven is given by T_n. It obeys Newton's Law of Cooling, so that

$$T_{n+1} - T_n = -k(T_n - 20),$$

where $n \geq 0$, $n \in \mathbb{Z}$ and $k \in \mathbb{R}$. When removed from the oven the chicken is at a temperature of 160°, i.e. $T_0 = 160$.

(i) Solve this difference equation to find T_n as a function of n and k.

(ii) After 5 minutes the chicken has cooled down to 98°. Show that $k = 0.11$.

(iii) Newton's Law of Cooling can instead be modelled using a differential equation. Write down a differential equation for $\frac{dT}{dn}$, the rate of change of the temperature of the chicken with respect to n (measured in minutes).

(iv) A tray of hot potatoes are also taken out of the oven at the same time as the hot chicken. They are initially at the same temperature of 160°. Their temperature at time n, measured in minutes, is given by the difference equation

$$P_{n+1} - P_n = -c(P_n - 20)$$

where $n \geq 0$, $n \in \mathbb{Z}$ and $c \in \mathbb{R}$.

It is found that, after 5 minutes, the temperature of the potatoes is lower than that of the chicken.

Is c less than or greater than k? Justify your answer.

Question — 50 marks, 5/10 difficulty. There is a referendum coming up in Ireland. There are only two types of voters, Yes voters and No voters. Y_n and N_n represents the proportion of Yes voters and No voters respectively, n days after the referendum was announced. Every day 5% of Yes voters turn into No voters, and 3% of No voters turn into Yes voters. This gives the pair of difference equations

$$Y_{n+1} - Y_n = -0.05Y_n + 0.03N_n,$$
$$N_{n+1} - N_n = -0.03N_n + 0.05Y_n,$$

for $n \geq 0$, $n \in \mathbb{Z}$. Initially 50% of voters are Yes voters and 50% of voters are No voters, i.e. $Y_0 = 0.5$ and $N_0 = 0.5$.

(i) Use the difference equations to show that

$$Y_{n+1} + N_{n+1} = Y_n + N_n$$

for all $n \geq 0$, and therefore that

$$Y_n + N_n = 1$$

for all $n \geq 0$.

(ii) Use the equation $Y_n + N_n = 1$ to show that

$$Y_n = 0.92Y_n + 0.03.$$

(iii) Solve the difference equation with initial condition

$$Y_n = 0.92Y_n + 0.03, \quad Y_0 = 0.5$$

to find Y_n in terms of n.

(iv) Use your answer to part (iii) and $Y_n + N_n = 1$ to find N_n in terms of n.

(v) If the referendum was held a long time into the future, and this model is correct, what percentage of voters would be No voters in the long run?

Question — 50 marks, 9/10 difficulty. An epidemiologist is studying the outbreak of a new disease among a population of people. He separates the people into Susceptible (S), Infected (I), and Recovered (R). Every month 4% of susceptible people become infected, and 19% of infected people recover. People are only considered recovered for one month until they are susceptible to getting the disease again.

The number of susceptible, infected and recovered people n months after the outbreak are given by S_n, I_n and R_n, measured in millions, which satisfy the following system of difference equations.

$$\begin{aligned} S_{n+1} &= 0.96S_n + R_n, \\ I_{n+1} &= 0.81I_n + 0.04S_n, \\ R_{n+1} &= 0.19I_n, \end{aligned}$$

for $n \geq 0$, $n \in \mathbb{Z}$. Initially there are only 5.94 million healthy, susceptible people, so $S_0 = 5.94$, $I_0 = 0$, $R_0 = 0$.

(i) Calculate S_1, I_1 and R_1.

(ii) Use the difference equations to show that

$$S_{n+1} + I_{n+1} + R_{n+1} = S_n + I_n + R_n,$$

for all $n \geq 0$, and therefore show that

$$S_n + I_n + R_n = 5.94$$

for all n.

(iii) Use $S_n + I_n + R_n = 5.94$ and the system of difference equations to show that

$$I_{n+2} - 0.77I_{n+1} + 0.0076I_n = 0.2376.$$

(iv) Use this difference equation, $I_0 = 0$ and the value of I_1 found in part (i) to find I_n as a function of n.

(v) In the long run, how many people, in millions, are susceptible, infected and recovered from the disease?

Question — 20 marks, 3/10 difficulty. The probability that it rains on any given day depends on whether or not it rained the previous day. For any given day, if it rained the previous day, there is a 40% chance it will rain that day. On the other hand, if it didn't rain the previous day, the probability it will rain that day is 20%.

If p_n is the probability that it rains on day n, then p_n satisfies the difference equation

$$p_n = 0.4p_n + 0.2(1 - p_n).$$

It rained on day 0, so $p_0 = 1$.

(i) Calculate p_1 and p_2.

(ii) Solve the difference equation to find p_n in terms of n.

(iii) Calculate p_{10}.

Question — 25 marks, 4/10 difficulty. Aoife takes out a €10,000 loan from the bank. She is charged a monthly interest of 1%, and makes increasing monthly repayments. In the first month, her repayment is A, and it increases by 0.4% each month. If M_n is the amount of money Aoife owes n months after taking out the loan, then

$$M_{n+1} = 1.01M_n - A(1.004)^n$$

for $n \geq 0$, $n \in \mathbb{Z}$ and $M_0 = 10,000$.

(i) Solve this difference equation to find M_n in terms of A and n.

(ii) Aoife plans on paying off this loan in 5 years (60 months) so that $M_{60} = 0$. Find A to the nearest cent.

Question — 50 marks, 8/10 difficulty. Charlie is studying a species of ant in an ant colony. He notices that each month, the number of surviving births in the ant colony is equal to twice the population at the beginning of the month. However he also notices that 29% of all ants aged one month or older at the beginning of the month die during that month.

If A_n is the number of ants in Charlie's ant colony in month n, then

$$A_{n+2} - A_{n+1} = 2A_{n+1} - 0.29A_n$$

for $n \geq 0$, $n \in \mathbb{Z}$. Originally Charlie has 70 ants in the ant colony, all less than one month old so $A_0 = 70$ and $A_1 = 210$.

(i) Calculate A_1 and A_2.

(ii) Solve this difference equation to find A_n as a function of n.

(iii) Calculate A_{10}.

(iv) Starting at the end of the second month (when $n = 2$) Charlie plans on removing ants from the colony to sell. He will initially remove 90 ants from the ant colony, and that amount will increase by 10% each month after that. If B_n is the number of ants in this colony under this model, then

$$B_{n+2} - B_{n+1} = 2B_{n+1} - 0.29B_n - 90(1.1)^n$$

for $n \geq 0$, $n \in \mathbb{Z}$, with $B_0 = 70$, $B_1 = 210$.

Calculate B_2.

(v) Solve this difference equation to find B_n as a function of n.

(vi) Calculate B_{10}.

(vii) In a third model, Charlie instead decides, starting at the end of the second month, to remove 50% of the number of ants that were present in the previous month.

If C_n is the number of ants present in the colony in month n under this model, write down a difference equation relating C_{n+2}, C_{n+1} and C_n.

Question — 50 marks, 7/10 difficulty. A car of mass 1000 kg is travelling on a smooth horizontal road. The engine provides a constant force of 3,000 N, and there is a resistance to motion that is proportional to its speed, with constant of proportionality equal to 20.

(i) If $v(t)$ is the speed of the car at time t (measured in seconds), show that

$$\frac{dv}{dt} = 20 - 0.05v.$$

(ii) If the car starts at rest, calculate the speed of the car after 1 minute.

(iii) It is instead decided to model the motion of the car using a difference equation. Let v_n be the speed of the car n seconds after it starts moving.

Write down a difference equation expressing v_{n+1} in terms of v_n.

(iv) If the car starts at rest, i.e. $v_0 = 0$, solve this difference equation to find v_n as a function of n.

(v) Find the speed of the car after 1 minute using the difference equation model.

(vi) Using either model, calculate the top speed of the car.