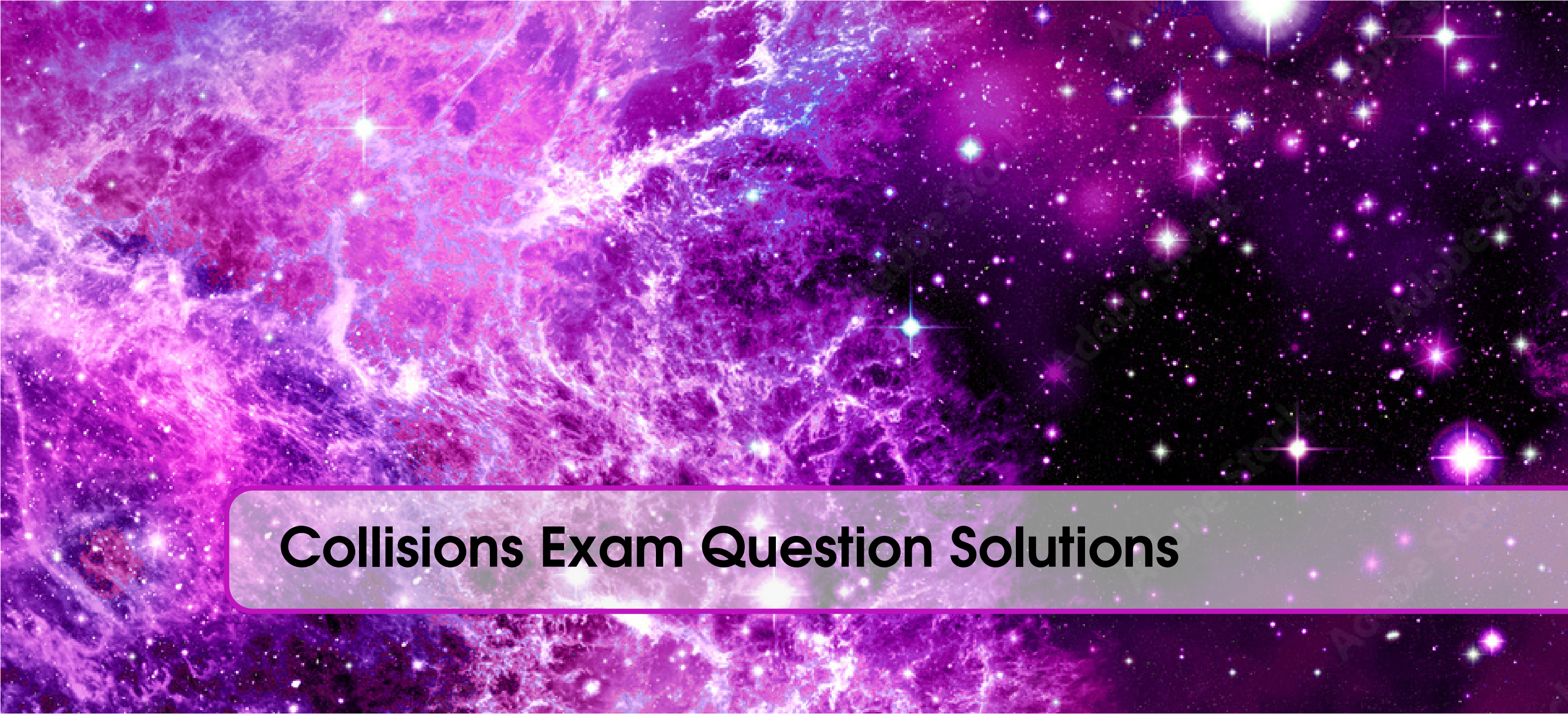
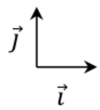


Collisions Exam Question Solutions

Note These exam questions are answered in reverse chronological order as they appear in exam papers; 2023 paper, Sample paper, 2022 (deferred), 2022, and so on back to 2015. They are answered in the style described in my notes. As the collisions portion of the syllabus is unchanged no questions from old syllabus exam papers have been excluded.

Question — 2023 Q2.

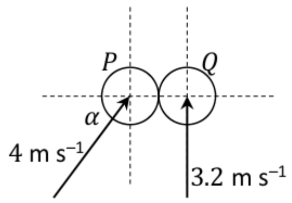
(b) Two smooth spheres, P and Q, have equal radius and are of mass m and $2m$ respectively. P and Q collide obliquely. The line joining their centres at the point of impact lies along the $\vec{i}$ axis.



Before the collision, sphere P moves with a velocity of 4 m s^{-1} at an angle α with the $\vec{i}$ axis, where $\sin\alpha = \frac{4}{5}$.

Before the collision, sphere Q moves with a velocity of 3.2 m s^{-1} perpendicular to the $\vec{i}$ axis.

The coefficient of restitution between the spheres is e, where $0 \le e \le 1$.



Calculate, in terms of e, the velocity of each sphere immediately after they collide

---

If $\sin\alpha = \frac{4}{5}$ then we can draw a triangle to show that $\cos\alpha = \frac{3}{5}$, $\tan\alpha = \frac{4}{3}$. Then the initial velocity of P satisfies

$$u_1 = 4\cos\alpha\,\vec{i} + 4\sin\alpha\,\vec{j}$$
$$= \frac{12}{5}\,\vec{i} + \frac{16}{5}\,\vec{j}.$$

Then we have the following information about the collision.

Before	Mass	After
$u_1 = \frac{12}{5}\vec{i} + \frac{16}{5}\vec{j}$	$m_1 = m$	$v_1 = p\,\vec{i} + \frac{16}{5}\vec{j}$
$u_2 = \frac{16}{5}\vec{j}$	$m_2 = 2m$	$v_2 = q\,\vec{i} + \frac{16}{5}\vec{j}$

$$\begin{aligned} m_1u_1 + m_2u_2 &= m_1v_1 + m_2v_2 \\ \Rightarrow m\left(\frac{12}{5}\right) + 2m(0) &= mp + 2mq \\ \Rightarrow \frac{12}{5} &= p + 2q. \end{aligned}$$

$$\begin{aligned} v_1 - v_2 &= -e(u_1 - u_2) \\ \Rightarrow p - q &= -e\left(\frac{12}{5} - 0\right) \\ &= -\frac{12}{5}e. \end{aligned}$$

Substituting out p from the second equation using the first,

$$\begin{aligned} \frac{12}{5} &= p + 2q \\ \Rightarrow \frac{12}{5} - 2q &= p, \end{aligned}$$

so that

$$\begin{aligned} p - q &= -\frac{12}{5}e \\ \Rightarrow \frac{12}{5} - 2q - q &= -\frac{12}{5}e \\ \Rightarrow -3q &= -\frac{12}{5} - \frac{12}{5}e \\ \Rightarrow q &= \frac{4}{5} + \frac{4}{5}e \\ \Rightarrow p &= \frac{12}{5} - 2\left(\frac{4}{5} + \frac{4}{5}e\right) \\ &= \frac{12}{5} - \frac{8}{5} - \frac{8}{5}e \\ &= \frac{4}{5} - \frac{8}{5}e. \end{aligned}$$

Therefore the final velocities are

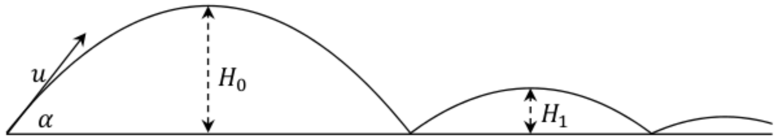
$$\begin{aligned} v_1 &= \left(\frac{4}{5} - \frac{8}{5}e\right)\vec{i} + \frac{16}{5}\vec{j} \\ v_2 &= \left(\frac{4}{5} + \frac{4}{5}e\right)\vec{i} + \frac{16}{5}\vec{j} \end{aligned}$$

Question — Sample Q4.

Question 4

(a) A ball is projected from a point on horizontal ground, with initial speed u and at an angle α to the horizontal. The ball reaches a maximum height of H_0 above the horizontal.

Upon landing, the ball bounces with a maximum height of H_1.

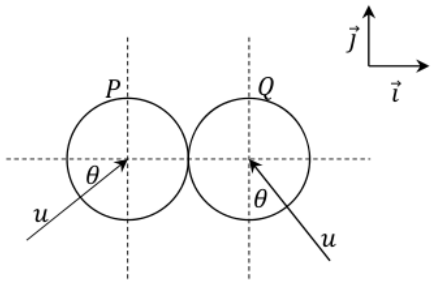


The coefficient of restitution between the ball and the ground is e.

(i) Calculate $\frac{H_0}{H_1}$.

(ii) The ball continues bouncing. Find an expression (in terms of e and H_0) for H_5, the maximum height of the ball after it lands on the ground for the fifth time.

(b) Two identical smooth spheres, P and Q, each moving with speed u, collide obliquely. The line joining their centres at the point of impact is along the $\vec{\imath}$ axis.



Before the collision, the velocity of sphere P makes an angle θ with the $\vec{\imath}$ axis and the velocity of sphere Q makes an angle θ with the $\vec{\jmath}$ axis, as shown in the diagram.

The coefficient of restitution between the spheres is e, where $0 \le e \le 1$.

After the collision sphere Q moves off parallel to the $\vec{\jmath}$ axis.

(i) Show that $e = \frac{\tan\theta - 1}{\tan\theta + 1}$.

(ii) If 25% of the spheres' total kinetic energy is lost during the collision, calculate θ and e.

(a)

Using (u, θ)**:**

(i) First, we can find H_0 easily enough.

x-axis	y-axis
$u_x = u\cos\alpha$	$u_y = u\sin\alpha$
$v_x = u\cos\alpha$	$v_y = 0$
$a_x = 0$	$a_y = -g$
$s_x =$	$s_y = H_0$
$t_x =$	$t_y = .$

$$
\begin{aligned}
v_y^2 &= u_y^2 + 2a_y s_y \\
\Rightarrow 0 &= u^2 \sin^2\alpha - 2gH_0 \\
\Rightarrow 2gH_0 &= u^2 \sin^2\alpha \\
\Rightarrow H_0 &= \frac{u^2 \sin^2\alpha}{2g}.
\end{aligned}
$$

We have the following UVAST array representing the two journeys, ending when the object is at height H_1.

First Part		Second Part		Extra Equations
x-axis	y-axis	x-axis	y-axis	
$u_{x1} = u\cos\alpha$	$u_{y1} = u\sin\alpha$	$u_{x2} = u\cos\alpha$	$u_{y2} =$	
$v_{x1} = u\cos\alpha$	$v_{y1} =$	$v_{x2} = u\cos\alpha$	$v_{y2} = 0$	$u_{y2} = -ev_{y1}$
$a_{x1} = 0$	$a_{y1} = -g$	$a_{x2} = 0$	$a_{y2} = -g$	
$s_{x1} =$	$s_{y1} = 0$	$s_{x2} =$	$s_{y2} = H_1$	
$t_{x1} = t_1$	$t_{y1} = t_1$	$t_{x2} = t_2$	$t_{y2} = t_2$	

First,

$$
\begin{aligned}
s_{y1} &= \left(\frac{u_{y1} + v_{y1}}{2}\right) t_{y1} \\
\Rightarrow 0 &= \left(\frac{u\sin\theta + v_{y1}}{2}\right) t_1 \\
\Rightarrow 0 &= u\sin\theta + v_{y1} \\
\Rightarrow -u\sin\theta &= v_{y1}.
\end{aligned}
$$

Then

$$
\begin{aligned}
u_{y2} &= -ev_{y1} \\
&= eu\sin\theta.
\end{aligned}
$$

Finally

$$
\begin{aligned}
v_{y2}^2 &= u_{y2}^2 + a_{y2} s_{y2} \\
\Rightarrow 0 &= e^2 u^2 \sin^2\alpha - 2gH_1 \\
\Rightarrow 2gH_1 &= e^2 u^2 \sin^2\alpha \\
\Rightarrow H_1 &= \frac{e^2 u^2 \sin^2\alpha}{2g}.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\frac{H_1}{H_0} &= \frac{\frac{e^2 u^2 \sin^2\alpha}{2g}}{\frac{u^2 \sin^2\alpha}{2g}} \\
&= e^2.
\end{aligned}
$$

(ii) H_n is a geometric sequence with common ratio e^2 and first term ($n = 1$ term) $H_1 = e^2H_0$. Therefore

$$\begin{aligned} H_5 &= e^2H_0(e^2)^4 \\ &= e^{10}H_0. \end{aligned}$$

Using (p,q):

(i) First, we can find H_0 easily enough.

x-axis	y-axis
$u_x = p$	$u_y = q$
$v_x = p$	$v_y = 0$
$a_x = 0$	$a_y = -g$
$s_x =$	$s_y = H_0$
$t_x =$	$t_y = .$

$$\begin{aligned} v_y^2 &= u_y^2 + 2a_ys_y \\ \Rightarrow 0 &= q^2 - 2gH_0 \\ \Rightarrow 2gH_0 &= q^2 \\ \Rightarrow H_0 &= \frac{q^2}{2g}. \end{aligned}$$

We have the following UVAST array representing the two journeys, ending when the object is at height H_1.

First Part		Second Part		Extra Equations
x-axis	y-axis	x-axis	y-axis	
$u_{x1} = p$	$u_{y1} = q$	$u_{x2} = p$	$u_{y2} =$	
$v_{x1} = p$	$v_{y1} =$	$v_{x2} = p$	$v_{y2} = 0$	$u_{y2} = -ev_{y1}$
$a_{x1} = 0$	$a_{y1} = -g$	$a_{x2} = 0$	$a_{y2} = -g$	
$s_{x1} =$	$s_{y1} = 0$	$s_{x2} =$	$s_{y2} = H_1$	
$t_{x1} = t_1$	$t_{y1} = t_1$	$t_{x2} = t_2$	$t_{y2} = t_2$	

First,

$$\begin{aligned} s_{y1} &= \left(\frac{u_{y1} + v_{y1}}{2}\right)t_{y1} \\ \Rightarrow 0 &= \left(\frac{q + v_{y1}}{2}\right)t_1 \\ \Rightarrow 0 &= q + v_{y1} \\ \Rightarrow -q &= v_{y1}. \end{aligned}$$

Then

$$\begin{aligned} u_{y2} &= -ev_{y1} \\ &= eq. \end{aligned}$$

Finally

$$\begin{aligned} v_{y2}^2 &= u_{y2}^2 + a_{y2}s_{y2} \\ \Rightarrow 0 &= e^2q^2 - 2gH_1 \\ \Rightarrow 2gH_1 &= e^2q^2 \\ \Rightarrow H_1 &= \frac{e^2q^2}{2g}. \end{aligned}$$

Therefore

$$\begin{aligned} \frac{H_1}{H_0} &= \frac{\frac{e^2q^2}{2g}}{\frac{q^2}{2g}} \\ &= e^2. \end{aligned}$$

(ii) H_n is a geometric sequence with common ratio e^2 and first term ($n=1$ term) $H_1 = e^2H_0$. Therefore

$$\begin{aligned} H_5 &= e^2H_0(e^2)^4 \\ &= e^{10}H_0. \end{aligned}$$

(b) (i) We have the following initial information. Notice how v_2 has no $\vec{i}$ component as we were told it moves purely in the $\vec{j}$ direction after collision.

Before	Mass	After
$u_1 = u\cos\theta\,\vec{i} + u\sin\theta\,\vec{j}$	$m_1 = M$	$v_1 = p\,\vec{i} + u\sin\theta\,\vec{j}$
$u_2 = -u\sin\theta\,\vec{i} + u\cos\theta\,\vec{j}$	$m_2 = M$	$v_2 = 0\,\vec{i} + u\cos\theta\,\vec{j}$

Let's apply our standard two equations to the $\vec{i}$ components of the initial and final velocities.

$$\begin{aligned} m_1u_1 + m_2u_2 &= m_1v_1 + m_2v_2 \\ \Rightarrow Mu\cos\theta - Mu\sin\theta &= Mp \\ \Rightarrow u\cos\theta - u\sin\theta &= p. \end{aligned}$$

$$\begin{aligned} v_1 - v_2 &= -e(u_1 - u_2) \\ \Rightarrow p - 0 &= -e(u\cos\theta + u\sin\theta). \end{aligned}$$

Letting our expressions for p be equal,

$$\begin{aligned} u\cos\theta - u\sin\theta &= -e(u\cos\theta + u\sin\theta) \\ \Rightarrow \frac{u\sin\theta - u\cos\theta}{u\cos\theta + u\sin\theta} &= e. \end{aligned}$$

Dividing by $\cos\theta$ (so that we can introduce $\tan\theta$) and cancelling the u,

$$\frac{\frac{\sin\theta}{\cos\theta}-1}{1+\frac{\sin\theta}{\cos\theta}}=e$$
$$\Rightarrow \frac{\tan\theta-1}{1+\tan\theta}=e.$$

(ii) Using $p=u\cos\theta-u\sin\theta$ (as it's more simple than the other version of p that uses e) and calculating the kinetic energies before and after we have

$$\begin{aligned}
\text{K.E.}_{\text{Before}} &= \frac{Mu^2}{2}+\frac{Mu^2}{2}\\
&= Mu^2.\\
\text{K.E.}_{\text{After}} &= \frac{M\sqrt{(u\cos\theta-u\sin\theta)^2+(u\sin\theta)^2}^2}{2}+\frac{M(u\cos\theta)^2}{2}\\
&= \frac{M\left(u^2\cos^2\theta-2u^2\cos\theta\sin\theta+u^2\sin^2\theta+u^2\sin^2\theta\right)}{2}+\frac{Mu^2\cos^2\theta}{2}\\
&= \frac{M\left(2u^2\cos^2\theta-2u^2\cos\theta\sin\theta+2u^2\sin^2\theta\right)}{2}\\
&= Mu^2\left(\cos^2\theta-\cos\theta\sin\theta+\sin^2\theta\right).
\end{aligned}$$

If 25% of the kinetic energy is lost then 75% is remaining and so

$$\begin{aligned}
\frac{\text{K.E.}_{\text{After}}}{\text{K.E.}_{\text{Before}}} &= 0.75\\
\Rightarrow \cos^2\theta-\cos\theta\sin\theta+\sin^2\theta &= 0.75\\
\Rightarrow \left[\cos^2\theta+\sin^2\theta\right]-\cos\theta\sin\theta &= 0.75\\
\Rightarrow 1-\frac{1}{2}\sin(2\theta) &= 0.75\\
\Rightarrow \frac{1}{2} &= \sin(2\theta)\\
\Rightarrow 30^\circ, 150^\circ &= 2\theta\\
\Rightarrow 15^\circ, 75^\circ &= \theta.
\end{aligned}$$

Finally,

$$\begin{aligned}
e &= \frac{\tan 15^\circ-1}{1+\tan 15^\circ}, \frac{\tan 75^\circ-1}{1+\tan 75^\circ}\\
&= -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}.
\end{aligned}$$

As we need $e>0$, $e=\frac{1}{\sqrt{3}}$ and $\theta=75^\circ$.

Question — 2022 (Deferred) Q5.

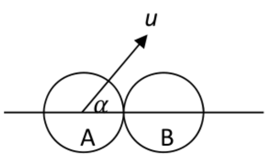
5. **(a)** A smooth sphere, P, of mass $3m$ collides directly with another smooth sphere, Q, of mass $5m$. P and Q are moving in opposite directions before impact with speeds $4u$ and $2u$ respectively. The coefficient of restitution for the collision is e.

(i) Find the speed of P and the speed of Q after impact in terms of u and e.

(ii) If P and Q are moving in the same direction after impact, show that $0 \leq e < \frac{1}{15}$.

(b) A smooth sphere, A, of mass m collides obliquely with another smooth sphere, B, of mass m.

Before impact, A is moving with speed u at an angle α to the line of centres of the spheres, where $0° < \alpha < 45°$.



B is at rest before the impact.

The coefficient of restitution for the collision is e.

(i) Find the speed of A and the speed of B after impact in terms of u, e and α.

(ii) Given that A is deflected through angle α because of the collision, show that $\tan^2 \alpha = e$.

(a) (i) Taking the direction of the P sphere as the positive direction,

$$\begin{aligned} m_1u_1 + m_2u_2 &= m_1v_1 + m_2v_2 \\ \Rightarrow 3m(4u) + 5m(-2u) &= 3mv_1 + 5mv_2 \\ \Rightarrow 2u &= 3v_1 + 5v_2. \end{aligned}$$

$$\begin{aligned} v_1 - v_2 &= -e(u_1 - u_2) \\ &= -e(4u - (-2u)) \\ &= -6eu \\ \Rightarrow v_1 &= v_2 - 6eu. \end{aligned}$$

Plugging this back into the first equation,

$$\begin{aligned} 2u &= 3(v_2 - 6eu) + 5v_2 \\ &= 3v_2 - 18eu + 5v_2 \\ &= 8v_2 - 18eu \\ \Rightarrow 2u + 18eu &= 8v_2 \\ \Rightarrow \frac{u + 9eu}{4} &= v_2. \end{aligned}$$

Then

$$\begin{aligned} v_1 &= v_2 - 6eu \\ &= \frac{u+9eu}{4} - \frac{24eu}{4} \\ &= \frac{u-15eu}{4}. \end{aligned}$$

(ii) $e \geq 0$ always. As $v_2 > 0$, if P and Q are travelling in the same direction then

$$\begin{aligned} v_1 &> 0 \\ \Rightarrow \frac{u-15eu}{4} &> 0 \\ \Rightarrow u - 15eu &> 0 \\ \Rightarrow u &> 15eu \\ \Rightarrow \frac{1}{15} &> e. \end{aligned}$$

(b) (i)

Before	Mass	After
$u_1 = u\cos\alpha\,\vec{i} + u\sin\alpha\,\vec{j}$	$m_1 = m$	$v_1 = p\,\vec{i} + u\sin\alpha\,\vec{j}$
$u_2 = 0$	$m_2 = m$	$v_2 = q\,\vec{i}$

$$\begin{aligned} m_1u_1 + m_2u_2 &= m_1v_1 + m_2v_2 \\ \Rightarrow m(u\cos\alpha) + m(0) &= mp + mq \\ \Rightarrow u\cos\alpha &= p + q. \end{aligned}$$

$$\begin{aligned} v_1 - v_2 &= -e(u_1 - u_2) \\ \Rightarrow p - q &= -e(u\cos\alpha - 0) \\ &= -eu\cos\alpha. \end{aligned}$$

$$\begin{aligned} p + q &= u\cos\alpha \\ (+)\ p - q &= -eu\cos\alpha \\ \hline 2p &= u\cos\alpha - eu\cos\alpha \\ &= u\cos\alpha(1-e) \\ \Rightarrow p &= \frac{u\cos\alpha(1-e)}{2}. \end{aligned}$$

$$
\begin{aligned}
p+q &= u\cos\alpha \\
(-)\ p-q &= -eu\cos\alpha \\
\hline
2q &= u\cos\alpha + eu\cos\alpha \\
&= u\cos\alpha(1+e) \\
\Rightarrow q &= \frac{u\cos\alpha(1+e)}{2}.
\end{aligned}
$$

Therefore the speed of A is

$$
\begin{aligned}
|v_1| &= \sqrt{\left(\frac{u\cos\alpha(1-e)}{2}\right)^2 + (u\sin\alpha)^2} \\
&= \sqrt{\frac{u^2\cos^2\alpha(1-2e+e^2)}{4} + u^2\sin^2\alpha}.
\end{aligned}
$$

As we weren't asked to show the speed in a specific form there's no need to try and simplify this. As B moves horizontally after collision the speed of B is simply

$$
|v_2| = \frac{u\cos\alpha(1+e)}{2}.
$$

(ii) The slope of u_1 is $\tan\alpha$. The slope of v_2 is

$$
\begin{aligned}
\frac{u\sin\alpha}{\frac{u\cos\alpha(1-e)}{2}} &= \frac{2u\sin\alpha}{u\cos\alpha(1-e)} \\
&= \frac{2\tan\alpha}{1-e}.
\end{aligned}
$$

Therefore the angle of deflection, $\tan\alpha$, is also equal to

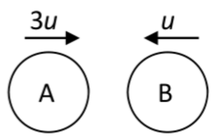
$$
\begin{aligned}
\tan\alpha &= \pm\frac{\tan\alpha - \frac{2\tan\alpha}{1-e}}{1+\tan\alpha\frac{2\tan\alpha}{1-e}} \\
&= \pm\frac{\tan\alpha(1-e) - 2\tan\alpha}{1-e+2\tan^2\alpha} \\
&= \pm\frac{-\tan\alpha - e\tan\alpha}{1-e+2\tan^2\alpha}
\end{aligned}
$$

The numerator is negative and the denominator is positive, and as $\tan\alpha > 0$ the $\pm = -$ so that

$$
\begin{aligned}
\tan\alpha &= \frac{\tan\alpha + e\tan\alpha}{1-e+2\tan^2\alpha} \\
\Rightarrow 1 &= \frac{1+e}{1-e+2\tan^2\alpha} \\
\Rightarrow 1-e+2\tan^2\alpha &= 1+e \\
\Rightarrow 2\tan^2\alpha &= 2e \\
\Rightarrow \tan^2\alpha &= e.
\end{aligned}
$$

Question — 2022 Q5.

(a) A smooth sphere A of mass $2m$, moving with speed $3u$ on a smooth horizontal table collides directly with a smooth sphere B of mass m, moving in the opposite direction with speed u.



The coefficient of restitution between A and B is e.

Find, in terms of u and e,

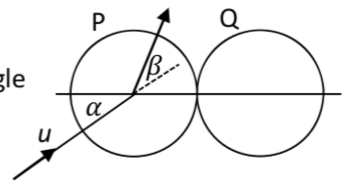
(i) the speed of each sphere after the collision

(ii) the magnitude of the impulse imparted to B due to the collision.

The loss of the kinetic energy due to the collision is $kmu^2(1-e^2)$.

(iii) Find the value of k.

(b) A smooth sphere P has mass m and speed u. It collides obliquely with a smooth sphere Q, of mass m, which is at rest. Before the collision, the direction of P makes an angle α with the line of centres, as shown in the diagram.



The coefficient of restitution between the spheres is $\frac{1}{3}$.

During the impact the direction of motion of P is turned through an angle β.

Show that $\tan\beta = \frac{2\tan\alpha}{1+3\tan^2\alpha}$.

(a) (i) If u_1, m_1, v_1 correspond to sphere A and u_2, m_2, v_2 correspond to sphere B, then using the principle of conservation of momentum,

$$\begin{aligned} m_1u_1 + m_2u_2 &= m_1v_1 + m_2v_2 \\ 2m(3u) + m(-u) &= 2mv_1 + mv_2 \\ \Rightarrow 5u &= 2v_1 + v_2. \end{aligned}$$

Using the coefficient of restitution equation,

$$\begin{aligned} v_1 - v_2 &= -e(u_1 - u_2) \\ &= -4eu. \end{aligned}$$

Solving the system of equations

$$\begin{aligned} 5u &= 2v_1 + v_2, \\ -4eu &= v_1 - v_2 \end{aligned}$$

in v_1, v_2 by solving for v_2 in the first equation and substituting it into the second,

$$\begin{aligned} 5u - 2v_1 &= v_2 \\ \Rightarrow -4eu &= v_1 - (5u - 2v_1) \\ &= v_1 - 5u + 2v_1 \\ \Rightarrow 5u - 4eu &= 3v_1 \\ \Rightarrow \frac{5u-4eu}{3} &= v_1 \\ \Rightarrow v_2 &= 5u - 2\left(\frac{5u-4eu}{3}\right) \\ &= \frac{15u}{3} - \frac{10u-8eu}{3} \\ &= \frac{5u+8eu}{3}. \end{aligned}$$

Note that as $0 \le e \le 1$ we can tell that v_1, v_2 are positive and so are the speeds and not just the velocities of the spheres.

(ii) The magnitude of the impulse imparted to B is given by

$$\begin{aligned} I &= m_2(v_2 - u_2) \\ &= m\left(\frac{5u+8eu}{3} - (-u)\right) \\ &= m\frac{8u+8eu}{3}. \end{aligned}$$

(iii) Calculating the loss of kinetic energy ourselves,

$$\begin{aligned} \text{K. E.}_{\text{Before}} &= \frac{m_1u_1^2}{2} + \frac{m_2u_2^2}{2} \\ &= \frac{2m(3u)^2}{2} + \frac{m(-u)^2}{2} \\ &= \frac{18mu^2}{2} + \frac{mu^2}{2} \\ &= \frac{19mu^2}{2}. \\ \text{K. E.}_{\text{After}} &= \frac{m_1v_1^2}{2} + \frac{m_2v_2^2}{2} \\ &= \frac{2m\left(\frac{5u-4eu}{3}\right)^2}{2} + \frac{m\left(\frac{5u+8eu}{3}\right)^2}{2} \\ &= \frac{2m\frac{25u^2-40eu^2+16e^2u^2}{9}}{2} + \frac{m\frac{25u^2+80eu^2+64e^2u^2}{9}}{2} \\ &= \frac{50mu^2 - 80emu^2 + 32e^2mu^2}{18} + \frac{25mu^2 + 80emu^2 + 64e^2mu^2}{18} \\ &= \frac{75mu^2 + 96e^2mu^2}{18} \\ &= \frac{25mu^2 + 32e^2mu^2}{6}. \end{aligned}$$

Therefore the loss in kinetic energy is given by

$$\begin{aligned}\frac{19mu^2}{2}-\frac{25mu^2+32e^2mu^2}{6}&=\frac{57mu^2}{6}-\frac{25mu^2+32e^2mu^2}{6}\\&=\frac{32mu^2-32e^2mu^2}{6}\\&=\frac{16mu^2-16e^2mu^2}{3}\\&=\frac{16}{3}(mu^2-e^2mu^2)\\&=\frac{16}{3}mu^2(1-e^2).\end{aligned}$$

Therefore

$$k=\frac{16}{3}.$$

(b) We have the following initial information.

Before	Mass	After
$u_1 = u\cos\alpha\,\vec{i}+u\sin\alpha\,\vec{j}$	$m_1 = m$	$v_1 = p\,\vec{i}+u\sin\alpha\,\vec{j}$
$u_2 = 0$	$m_2 = m$	$v_2 = q\,\vec{i}$

Let's apply our standard two equations to the $\vec{i}$ components of the initial and final velocities.

$$\begin{aligned}m_1u_1+m_2u_2&=m_1v_1+m_2v_2\\\Rightarrow mu\cos\alpha+0&=mp+mq\\\Rightarrow u\cos\alpha&=p+q.\end{aligned}$$

$$\begin{aligned}v_1-v_2&=-e(u_1-u_2)\\\Rightarrow p-q&=-\frac{1}{3}u\cos\alpha.\end{aligned}$$

We only care about p, so adding these equations

$$\begin{aligned}p+q&=u\cos\alpha\\(+)\ p-q&=-\frac{1}{3}\cos\alpha\\\hline 2p&=\frac{2}{3}u\cos\alpha\\\Rightarrow p&=\frac{1}{3}u\cos\alpha.\end{aligned}$$

See that the slope of $v_1=\frac{1}{3}u\cos\alpha\,\vec{i}+u\sin\alpha\,\vec{j}$ is

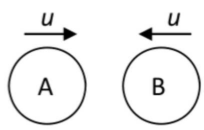
$$\frac{u\sin\alpha}{\frac{1}{3}u\cos\alpha}=3\tan\alpha.$$

The slope of u_1 is $\tan\alpha$. As both are positive β is acute and $\tan\beta$ is positive. Therefore

$$\begin{aligned}\tan\beta &= \pm\frac{\tan\alpha - 3\tan\alpha}{1+3\tan^2\alpha}\\ &= \pm\frac{-2\tan\alpha}{1+3\tan^2\alpha}\\ &= \frac{2\tan\alpha}{1+3\tan^2\alpha}.\end{aligned}$$

Question — 2021 Q5.

(a) A smooth sphere A of mass $4m$, moving with speed u on a smooth horizontal table collides directly with a smooth sphere B of mass m, moving in the opposite direction with speed u.



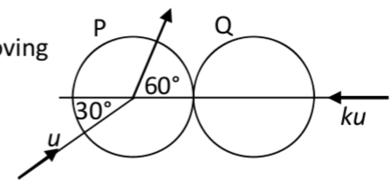
The coefficient of restitution between A and B is e.

(i) Find the speed, in terms of u and e, of each sphere after the collision.

The magnitude of the impulse on B due to the collision is T.

(ii) Show that $\frac{8mu}{5} \leq T \leq \frac{16mu}{5}$.

(b) A smooth sphere P has mass $2m$ and speed u. It collides obliquely with a smooth sphere Q of mass m which is moving with speed ku, as shown in the diagram.
Before the collision, the direction of P makes an angle of 30° to the line of centres. After the collision, the direction of P makes an angle of 60° to the line of centres.



The coefficient of restitution between the spheres is e.

(i) Show that $k = \frac{\sqrt{3}(1-e)}{2(1+e)}$.

(ii) Find the speed of Q immediately after the collision.

(a) (i) Taking the direction of sphere A as the positive direction,

$$\begin{aligned} m_1u_1 + m_2u_2 &= m_1v_1 + m_2v_2 \\ \Rightarrow 4m(u) + m(-u) &= 4mv_1 + mv_2 \\ \Rightarrow 3u &= 4v_1 + v_2. \end{aligned}$$

$$\begin{aligned} v_1 - v_2 &= -e(u_1 - u_2) \\ &= -e(u - (-u)) \\ &= -2eu. \end{aligned}$$

$$\begin{aligned} 4v_1 + v_2 &= 3u \\ (+)\ v_1 - v_2 &= -2eu \\ \hline 5v_1 &= 3u - 2eu \\ \Rightarrow v_1 &= \frac{3u - 2eu}{5}. \end{aligned}$$

Then

$$\begin{aligned} 4v_1 + v_2 &= 3u \\ \Rightarrow 4\frac{3u-2eu}{5} + v_2 &= 3u \\ \Rightarrow v_2 &= \frac{15u}{5} - \frac{12u-8eu}{5} \\ &= \frac{3u+8eu}{5}. \end{aligned}$$

(ii) The impulse imparted onto B is

$$\begin{aligned} T &= m_2(v_2 - u_2) \\ &= m\left(\frac{3u+8eu}{5} - (-u)\right) \\ &= m\left(\frac{8u+8eu}{5}\right). \end{aligned}$$

As $0 \le e \le 1$,

$$\begin{aligned} m\left(\frac{8u+8(0)u}{5}\right) \le T \le m\left(\frac{8u+8(1)u}{5}\right) \\ \Rightarrow \frac{8mu}{5} \le T \le \frac{16mu}{5}. \end{aligned}$$

(b) (i) Using the fact that $\cos 30^\circ = \frac{\sqrt{3}}{2}$, $\sin 30^\circ = \frac{1}{2}$, we have the following information about the collision.

Before	Mass	After
$u_1 = \frac{\sqrt{3}u}{2}\vec{i} + \frac{u}{2}\vec{j}$	$m_1 = 2m$	$v_1 = p\,\vec{i} + \frac{u}{2}\vec{j}$
$u_2 = -ku\,\vec{i}$	$m_2 = m$	$v_2 = q\,\vec{i}$

We also know that

$$\begin{aligned} \tan 60^\circ &= \frac{\frac{u}{2}}{p} \\ \Rightarrow \sqrt{3} &= \frac{u}{2p} \\ \Rightarrow 2\sqrt{3}p &= u \\ \Rightarrow p &= \frac{u}{2\sqrt{3}}. \end{aligned}$$

Then applying our usual two equations,

$$
\begin{aligned}
m_1u_1+m_2u_2 &= m_1v_1+m_2v_2\\
\Rightarrow 2m\left(\frac{\sqrt{3}u}{2}\right)+m(-ku) &= 2m\left(\frac{u}{2\sqrt{3}}\right)+mq\\
\Rightarrow \sqrt{3}u-ku &= \frac{u}{\sqrt{3}}+q\\
\Rightarrow \left(\sqrt{3}-k-\frac{1}{\sqrt{3}}\right)u &= q\\
\Rightarrow \left(\frac{2}{\sqrt{3}}-k\right)u &= q.
\end{aligned}
$$

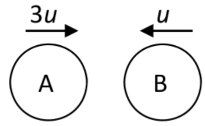
$$
\begin{aligned}
v_1-v_2 &= -e(u_1-u_2)\\
\Rightarrow \frac{u}{2\sqrt{3}}-\left(\frac{2}{\sqrt{3}}-k\right)u &= -e\left(\frac{\sqrt{3}u}{2}-(-ku)\right)\\
\Rightarrow -\frac{3u}{2\sqrt{3}}+ku &= -\frac{\sqrt{3}eu}{2}-eku\\
\Rightarrow -\frac{3}{2\sqrt{3}}+k &= -\frac{\sqrt{3}e}{2}-ek\\
\Rightarrow k+ek &= \frac{3}{2\sqrt{3}}-\frac{\sqrt{3}e}{2}\\
\Rightarrow k(1+e) &= \frac{3-\sqrt{3}(\sqrt{3}e)}{2\sqrt{3}}\\
\Rightarrow k &= \frac{3-3e}{2\sqrt{3}(1+e)}\\
&= \frac{3(1-e)}{2\sqrt{3}(1+e)}\\
&= \frac{\sqrt{3}(1-e)}{2(1+e)}.
\end{aligned}
$$

(ii) As Q moves horizontally after the collision, its speed is

$$
\begin{aligned}
v_2 &= \left(\frac{2}{\sqrt{3}}-\frac{\sqrt{3}(1-e)}{2(1+e)}\right)u\\
&= \left(\frac{2\sqrt{3}}{3}-\frac{\sqrt{3}(1-e)}{2(1+e)}\right)u\\
&= \left(\frac{4\sqrt{3}(1+e)}{6(1+e)}-\frac{3\sqrt{3}(1-e)}{6(1+e)}\right)u\\
&= \left(\frac{4\sqrt{3}+4\sqrt{3}e-3\sqrt{3}+3\sqrt{3}e}{6(1+e)}\right)u\\
&= \frac{\sqrt{3}+7\sqrt{3}e}{6(1+e)}u.
\end{aligned}
$$

Question — 2020 Q5.

(a) A smooth sphere A of mass m, moving with speed $3u$ on a smooth horizontal table collides directly with a smooth sphere B of mass $2m$, moving in the opposite direction with speed u.
The directions of motion of A and B are reversed by the collision.



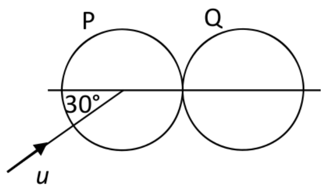
The coefficient of restitution between A and B is e.

(i) Find the speed, in terms of u and e, of each sphere after the collision.

Subsequently B hits a wall at right angles to the line of motion of A and B.
The coefficient of restitution between B and the wall is $\frac{1}{2}$.
After B rebounds from the wall there is a further collision between A and B.

(ii) Show that $\frac{1}{8} < e < \frac{1}{4}$.

(b) A smooth sphere P has mass m_1 and speed u.
It collides obliquely with a smooth sphere Q, of mass m_2, which is at rest.
Before the collision the direction of P makes an angle of 30° to the line of centres, as shown in the diagram.



The coefficient of restitution between the spheres is e.

Prove that P will turn through a right-angle if $4m_1 = (3e - 1)m_2$.

(a) (i) Taking the direction of the A sphere as the positive direction,

$$\begin{aligned} m_1u_1 + m_2u_2 &= m_1v_1 + m_2v_2 \\ \Rightarrow m(3u) + 2m(-u) &= mv_1 + 2mv_2 \\ \Rightarrow u &= v_1 + 2v_2. \end{aligned}$$

$$\begin{aligned} v_1 - v_2 &= -e(u_1 - u_2) \\ &= -e(3u - (-u)) \\ &= -4eu \\ \Rightarrow v_1 &= v_2 - 4eu. \end{aligned}$$

Plugging this into the first equation,

$$\begin{aligned} u &= v_2 - 4eu + 2v_2 \\ \Rightarrow u + 4eu &= 3v_2 \\ \Rightarrow \frac{u+4eu}{3} &= v_2 \\ \Rightarrow v_1 &= \frac{u+4eu}{3} - 4eu \\ &= \frac{u+4eu}{3} - \frac{12eu}{3} \\ &= \frac{u-8eu}{3}. \end{aligned}$$

As we are explicitly told that A changes direction we know $v_1 < 0$, so that the speed of A is

$$\begin{aligned}|v_1| &= -v_1 \\ &= \frac{8eu - u}{3}.\end{aligned}$$

(ii) Letting v_3 be the velocity of B after it hits the wall,

$$\begin{aligned}v_3 &= -\frac{1}{2}v_2 \\ &= -\frac{1}{2}\frac{u+4eu}{3} \\ &= -\frac{u+4eu}{6}.\end{aligned}$$

As both A and B change direction after the collision (given in question) we have $v_1 < 0$, $v_2 > 0$. $v_2 > 0$ is clear as $0 \leq e \leq 1$ and so is unhelpful, but

$$\begin{aligned}v_1 &< 0 \\ \Rightarrow \frac{u - 8eu}{3} &< 0 \\ \Rightarrow u - 8eu &< 0 \\ \Rightarrow u &< 8eu \\ \Rightarrow \frac{1}{8} &< e.\end{aligned}$$

As A and B collide again,

$$\begin{aligned}v_1 &> v_3 \\ \Rightarrow \frac{u-8eu}{3} &> -\frac{u+4eu}{6} \\ \Rightarrow 2u - 16eu &> -u - 4eu \\ \Rightarrow 3u &> 12eu \\ \Rightarrow \frac{1}{4} &> e.\end{aligned}$$

(b) Using the fact that $\cos 30^\circ = \frac{\sqrt{3}}{2}$, $\sin 30^\circ = \frac{1}{2}$, we have the following information about the collision.

Before	Mass	After
$u_1 = \frac{\sqrt{3}u}{2}\vec{i} + \frac{u}{2}\vec{j}$	$m_1 = m_1$	$v_1 = p\vec{i} + \frac{u}{2}\vec{j}$
$u_2 = 0$	$m_2 = m_2$	$v_2 = q\vec{i}$

Applying our usual two equations to the collision,

$$\begin{aligned}
m_1u_1 + m_2u_2 &= m_1v_1 + m_2v_2 \\
\Rightarrow m_1\left(\frac{\sqrt{3}u}{2}\right) + m_2(0) &= m_1p + m_2q \\
\Rightarrow \frac{\sqrt{3}um_1}{2} &= m_1p + m_2q.
\end{aligned}$$

$$\begin{aligned}
v_1 - v_2 &= -e(u_1 - u_2) \\
\Rightarrow p - q &= -e\left(\frac{\sqrt{3}u}{2} - 0\right) \\
&= -\frac{\sqrt{3}eu}{2}.
\end{aligned}$$

We only care about P and therefore p, so using the second equation to substitute q out of the first,

$$\begin{aligned}
p - q &= -\frac{\sqrt{3}eu}{2} \\
\Rightarrow p + \frac{\sqrt{3}eu}{2} &= q \\
\Rightarrow \frac{\sqrt{3}um_1}{2} &= m_1p + m_2\left(p + \frac{\sqrt{3}eu}{2}\right) \\
&= m_1p + m_2p + \frac{\sqrt{3}eum_2}{2} \\
\Rightarrow \frac{\sqrt{3}um_1}{2} - \frac{\sqrt{3}eum_2}{2} &= (m_1 + m_2)p \\
\Rightarrow \frac{\sqrt{3}u(m_1 - em_2)}{2(m_1 + m_2)} &= p.
\end{aligned}$$

The slope of the final velocity of P is therefore

$$\begin{aligned}
m_2 &= \frac{\frac{u}{2}}{\frac{\sqrt{3}u(m_1 - em_2)}{2(m_1 + m_2)}} \\
&= \frac{1}{\frac{\sqrt{3}(m_1 - em_2)}{(m_1 + m_2)}} \\
&= \frac{m_1 + m_2}{\sqrt{3}(m_1 - em_2)}.
\end{aligned}$$

The slope of the initial velocity of P is $\tan 30^\circ = \frac{1}{\sqrt{3}}$. If $4m_1 = (3e-1)m_2$ then the slope of the final velocity of P is

$$\begin{aligned}
\frac{m_1+m_2}{\sqrt{3}(m_1-em_2)} &= \frac{4m_1+4m_2}{\sqrt{3}(4m_1-4em_2)} \\
&= \frac{(3e-1)m_2+4m_2}{\sqrt{3}((3e-1)m_2-4em_2)} \\
&= \frac{3em_2-m_2+4m_2}{\sqrt{3}(3em_2-m_2-4em_2)} \\
&= \frac{3em_2+3m_2}{\sqrt{3}(-em_2-m_2)} \\
&= \frac{3e+3}{\sqrt{3}(-e-1)} \\
&= \frac{3(e+1)}{-\sqrt{3}(e+1)} \\
&= -\frac{3}{\sqrt{3}}.
\end{aligned}$$

As

$$\frac{1}{\sqrt{3}} \times -\frac{3}{\sqrt{3}} = -1$$

the slopes before and after collision are at right angles to each other.

Question — 2019 Q5.

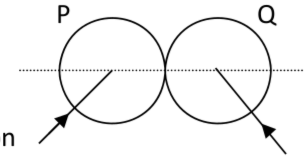
(a) A small smooth sphere A, of mass $3m$ moving with speed u, collides directly with a small smooth sphere B, of mass m moving with speed u in the opposite direction. The coefficient of restitution between the spheres is $\frac{1}{2}$.

(i) Find, in terms of u, the speed of each sphere after the collision.

After the collision B hits a smooth vertical wall which is perpendicular to the direction of motion of B. The coefficient of restitution between B and the wall is $\frac{2}{5}$.
The first collision between the spheres occurred at a distance 2 metres from the wall.
The spheres collide again 4 seconds after the first collision between them.

(ii) Find the value of u.

(b) A smooth sphere P, of mass $2m$, collides with a smooth sphere Q, of mass m.
The velocity of P is $3u\,\vec{i} + 4u\,\vec{j}$ and the velocity of Q is $-4u\,\vec{i} + 3u\,\vec{j}$, where $\vec{i}$ is along the line of centres at impact.



The coefficient of restitution between the spheres is $\frac{5}{7}$.

Find
(i) in terms of u, the speed of each sphere after the collision
(ii) the angle between the directions of P and Q after the collision.

(a) (i) Letting the direction of the A sphere be the positive direction,

$$\begin{aligned} m_1u_1 + m_2u_2 &= m_1v_1 + m_2v_2 \\ \Rightarrow 3m(u) + m(-u) &= 3mv_1 + mv_2 \\ \Rightarrow 2u &= 3v_1 + v_2. \end{aligned}$$

$$\begin{aligned} v_1 - v_2 &= -e(u_1 - u_2) \\ &= -\frac{1}{2}(u - (-u)) \\ &= -u. \end{aligned}$$

Adding these equations,

$$\begin{aligned} 3v_1 + v_2 &= 2u \\ (+)\ v_1 - v_2 &= -u \\ \hline 4v_1 \quad &= u \\ \Rightarrow v_1 &= \frac{u}{4} \\ \Rightarrow 2u &= 3\left(\frac{u}{4}\right) + v_2 \\ \Rightarrow \frac{5u}{4} &= v_2. \end{aligned}$$

(ii) It takes B

$$\frac{2}{\frac{5u}{4}} = \frac{8}{5u} \text{ seconds}$$

to hit the wall. In this time A travels

$$\frac{u}{4}\frac{8}{5u} = \frac{2}{5} \text{ m.}$$

If v_3 is the speed of B after collision (speed is easier to deal with here as this is becoming more of a linear motion problem) then

$$\begin{aligned} v_3 &= \frac{2}{5}v_2 \\ &= \frac{2}{5}\left(\frac{5u}{4}\right) \\ &= \frac{u}{2}. \end{aligned}$$

The spheres collide again $4 - \frac{8}{5u}$ seconds after B hits the wall. In this time the distance travelled between them is

$$\begin{aligned} \frac{u}{4}\left(4 - \frac{8}{5u}\right) + \frac{u}{2}\left(4 - \frac{8}{5u}\right) &= u - \frac{2}{5} + 2u - \frac{4}{5} \\ &= 3u - \frac{6}{5}. \end{aligned}$$

As this distance is $2 - \frac{2}{5} = \frac{8}{5}$ metres,

$$\begin{aligned} 3u - \frac{6}{5} &= \frac{8}{5} \\ \Rightarrow 3u &= \frac{14}{5} \\ \Rightarrow u &= \frac{14}{15} \text{ m/s.} \end{aligned}$$

(b) (i)

Before	Mass	After
$u_1 = 3u\,\vec{i} + 4u\,\vec{j}$	$m_1 = 2m$	$v_1 = p\,\vec{i} + 4u\,\vec{j}$
$u_2 = -4u\,\vec{i} + 3u\,\vec{j}$	$m_2 = m$	$v_2 = q\,\vec{i} + 3u\,\vec{j}$

$$\begin{aligned} m_1u_1 + m_2u_2 &= m_1v_1 + m_2v_2 \\ \Rightarrow 2m(3u) + m(-4u) &= 2mp + mq \\ \Rightarrow 2u &= 2p + q. \end{aligned}$$

$$\begin{aligned} v_1 - v_2 &= -e(u_1 - u_2) \\ \Rightarrow p - q &= -\frac{5}{7}(3u - (-4u)) \\ &= -5u. \end{aligned}$$

Adding these equations,

$$\begin{aligned} 2p + q &= 2u \\ (+)\ p - q &= -5u \\ \hline 3p \quad &= -3u \\ \Rightarrow p &= -u \\ \Rightarrow 2u &= 2(-u) + q \\ \Rightarrow 4u &= q. \end{aligned}$$

The speed of P is therefore

$$\begin{aligned} |v_1| &= \sqrt{(-u)^2 + (4u)^2} \\ &= \sqrt{u^2 + 16u^2} \\ &= \sqrt{17}u. \end{aligned}$$

The speed of Q is therefore

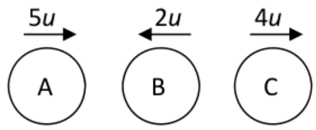
$$\begin{aligned} |v_1| &= \sqrt{(4u)^2 + (3u)^2} \\ &= \sqrt{16u^2 + 9u^2} \\ &= 5u. \end{aligned}$$

(ii) As the velocities are quite numerical and we already have the speeds it's easier to use the dot product formula than the angle between two lines formula. If θ is the angle between v_1 and v_2 then

$$\begin{aligned} \cos\theta &= \frac{v_1 \cdot v_2}{|v_1||v_2|} \\ &= \frac{-u(4u) + 4u(3u)}{\sqrt{17}u(5u)} \\ &= \frac{8u^2}{5\sqrt{17}u^2} \\ &= \frac{8}{5\sqrt{17}} \\ \Rightarrow \theta &= 67^\circ. \end{aligned}$$

Question — 2018 Q5.

(a) Three identical small smooth spheres A, B and C, each of mass m, lie in a straight line on a smooth horizontal surface with B between A and C.
Spheres A and B are projected towards each other with speeds $5u$ and $2u$ respectively, and at the same time C is projected along the line from B away from B with speed $4u$.
The coefficient of restitution between each pair of spheres is e.
After the collision between A and B there is a collision between B and C.

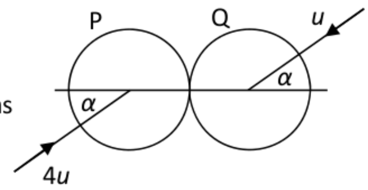


(i) Find, in terms of e and u, the speed of each sphere after the first collision.

(ii) Show $e > \frac{5}{7}$.

(iii) If $e = \frac{6}{7}$ show that B will not collide with A again.

(b) A small smooth sphere P, of mass $2m$, moving with speed $4u$, collides obliquely with an equal smooth sphere Q, of mass $3m$, moving with speed u.
Before the collision the spheres are moving in opposite directions, each making an angle α to the line of centres, as shown in the diagram.



The coefficient of restitution between the spheres is $\frac{1}{5}$.

(i) Find, in terms of u and α, the speed of each sphere after the collision.

After the collision the speed of P is twice the speed of Q.

(ii) Find the value of α.

(a) (i) As the spheres are identical we can assume they're all of mass m.

$$m_1u_1 + m_2u_2 = m_1v_1 + m_2v_2$$
$$\Rightarrow m(5u) + m(-2u) = mv_1 + mv_2$$
$$\Rightarrow 3u = v_1 + v_2.$$

$$v_1 - v_2 = -e(u_1 - u_2)$$
$$= -e(5u - (-2u))$$
$$= -7eu.$$

Adding these equations

$$\begin{aligned} v_1 + v_2 &= 3u \\ (+)\ v_1 - v_2 &= -7eu \end{aligned}$$

$$\begin{aligned} 2v_1 \qquad &= 3u - 7eu \\ \Rightarrow v_1 &= \frac{3u-7eu}{2} \\ \Rightarrow 3u &= \frac{3u+7eu}{2} + v_2 \\ \Rightarrow \frac{6u}{2} - \frac{3u-7eu}{2} &= v_2 \\ \Rightarrow \frac{3u+7eu}{2} &= v_2. \end{aligned}$$

(ii) Because there is a collision between B and C,

$$\begin{aligned} v_2 &> 4u \\ \Rightarrow \frac{3u+7eu}{2} &> 4u \\ \Rightarrow 3u+7eu &> 8u \\ \Rightarrow 7eu &> 5u \\ \Rightarrow e &> \frac{5}{7}. \end{aligned}$$

(iii) If $e = \frac{6}{7}$ then

$$\begin{aligned} v_1 &= \frac{3u - 7\left(\frac{6}{7}\right)u}{2} \\ &= \frac{3u-6u}{2} \\ &= -\frac{3u}{2}. \\ v_2 &= \frac{3u + 7\left(\frac{6}{7}\right)u}{2} \\ &= \frac{3u+6u}{2} \\ &= \frac{9u}{2}. \end{aligned}$$

Studying the collision between B and C and calling the final velocities v_3, v_4 respectively, our usual two collision equations give us

$$\begin{aligned} m\left(\frac{9u}{2}\right) + m(4u) &= mv_3 + mv_4 \\ \Rightarrow \frac{17u}{2} &= v_3 + v_4 \end{aligned}$$

and

$$
\begin{aligned}
v_3 - v_4 &= -\frac{6}{7}\left(\frac{9u}{2} - 4u\right) \\
&= -\frac{6}{7}\left(\frac{u}{2}\right) \\
\Rightarrow v_3 - v_4 &= -\frac{3u}{7}.
\end{aligned}
$$

Adding the equations (because we don't actually care about v_4)

$$
\begin{aligned}
v_3 + v_4 &= \frac{17u}{2} \\
(+)\ v_3 - v_4 &= -\frac{3u}{7} \\
\hline
2v_3 \quad &= \frac{113u}{14} \\
\Rightarrow v_3 &= \frac{113u}{28}.
\end{aligned}
$$

As $v_3 > v_1$ A and B don't collide again.

(b) (i)

Before	Mass	After
$u_1 = 4u\cos\alpha\,\vec{i} + 4u\sin\alpha\,\vec{j}$	$m_1 = 2m$	$v_1 = p\,\vec{i} + 4u\sin\alpha\,\vec{j}$
$u_2 = -u\cos\alpha\,\vec{i} - u\sin\alpha\,\vec{j}$	$m_2 = 3m$	$v_2 = q\,\vec{i} - u\sin\alpha\,\vec{j}$

Applying our usual two equations to the collisions,

$$
\begin{aligned}
m_1u_1 + m_2u_2 &= m_1v_1 + m_2v_2 \\
\Rightarrow 2m(4u\cos\alpha) + 3m(-u\sin\alpha) &= 2mv_1 + 3mv_2 \\
\Rightarrow 5u\cos\alpha &= 2v_1 + 3v_2.
\end{aligned}
$$

$$
\begin{aligned}
v_1 - v_2 &= -e(u_1 - u_2) \\
&= -\frac{1}{5}\left(4u\cos\alpha - (-u\cos\alpha)\right) \\
&= -u\cos\alpha \\
\Rightarrow v_1 &= v_2 - u\cos\alpha \\
\Rightarrow 5u\cos\alpha &= 2(v_2 - u\cos\alpha) + 3v_2 \\
&= 2v_2 - 2u\cos\alpha + 3v_2 \\
\Rightarrow 7u\cos\alpha &= 5v_2 \\
\Rightarrow \frac{7}{5}u\cos\alpha &= v_2 \\
\Rightarrow v_1 &= \frac{7}{5}u\cos\alpha - u\cos\alpha \\
&= \frac{2}{5}u\cos\alpha.
\end{aligned}
$$

The speed of P after the collision is therefore

$$|v_1| = \sqrt{\left(\frac{2}{5}u\cos\alpha\right)^2 + (4u\sin\alpha)^2}$$
$$= \sqrt{\frac{4}{25}u^2\cos^2\alpha + 16u^2\sin^2\alpha}.$$

The speed of Q after the collision is

$$|v_1| = \sqrt{\left(\frac{7}{5}u\cos\alpha\right)^2 + (-u\sin\alpha)^2}$$
$$= \sqrt{\frac{49}{25}u^2\cos^2\alpha + u^2\sin^2\alpha}.$$

As we were not asked for the speed in a specific form there is no need to attempt to simplify this further.

(ii)

$$\begin{aligned}
|v_1| &= 2|v_2| \\
\Rightarrow \sqrt{\frac{4}{25}u^2\cos^2\alpha + 16u^2\sin^2\alpha} &= 2\sqrt{\frac{49}{25}u^2\cos^2\alpha + u^2\sin^2\alpha} \\
\Rightarrow \frac{4}{25}u^2\cos^2\alpha + 16u^2\sin^2\alpha &= 4\left(\frac{49}{25}u^2\cos^2\alpha + u^2\sin^2\alpha\right) \\
&= \frac{196}{25}u^2\cos^2\alpha + 4u^2\sin^2\alpha \\
\Rightarrow 12u^2\sin^2\alpha &= \frac{192}{25}u^2\cos^2\alpha \\
\Rightarrow \frac{\sin^2\alpha}{\cos^2\alpha} &= \frac{16}{25} \\
\Rightarrow \tan^2\alpha &= \frac{16}{25} \\
\Rightarrow \tan\alpha &= \frac{4}{5} \\
\Rightarrow \alpha &= 39^\circ.
\end{aligned}$$

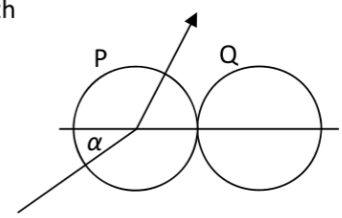
Question — 2017 Q5.

(a) A small smooth sphere A, of mass 1·5 kg, moving with speed 6 m s^{-1}, collides directly with a small smooth sphere B, of mass m kg, which is at rest.
After the collision the spheres move in opposite directions with speeds v and $2v$, respectively.
80% of the kinetic energy lost by A as a result of the collision is transferred to B.
The coefficient of restitution between the spheres is e.

Find **(i)** the value of v
(ii) the value of e.

(b) A small smooth sphere P, of mass $3m$, collides obliquely with a small smooth sphere Q, of mass $7m$, which is at rest.

Before the collision the velocity of P makes an angle α with the line joining the centres of the spheres.
After the collision the speed of Q is v.
The coefficient of restitution between the spheres is $\frac{2}{7}$.



(i) Find, in terms of v and α, the **speed** of P before the collision.

(ii) If $\alpha = 30°$ find the angle through which the direction of motion of P is deflected as a result of the collision.

(a) (i) Taking the initial direction of A as the positive direction, it is clear that B is travelling in this direction after collision. Then

$$m_1u_1 + m_2u_2 = m_1v_1 + m_2v_2$$
$$\Rightarrow 1.5(6) + m(0) = 1.5(-v) + m(2v)$$
$$\Rightarrow 9 = -1.5v + 2mv.$$

$$v_1 - v_2 = -e(u_1 - u_2)$$
$$\Rightarrow -v - 2v = -e(6)$$
$$\Rightarrow -3v = -6e$$
$$\Rightarrow \frac{v}{2} = e.$$

If 80% of the kinetic energy lost by A is transferred to B, then let's explore that. The kinetic energy lost by A is given by

$$\frac{m_1u_1^2}{2} - \frac{m_1v_1^2}{2} = \frac{1.5(6)^2}{2} - \frac{1.5v^2}{2}$$
$$= 27 - \frac{3}{4}v^2.$$

The kinetic energy gained by B is simply

$$\begin{aligned}\frac{m_2 v_2^2}{2} &= \frac{m(2v)^2}{2} \\ &= 2mv^2.\end{aligned}$$

Therefore if 80% of the first amount of energy is transferred to the second,

$$\begin{aligned}\frac{4}{5}\left(27 - \frac{3}{4}v^2\right) &= 2mv^2 \\ \Rightarrow \frac{108}{5} - \frac{3}{5}v^2 &= 2mv^2 \\ \Rightarrow \frac{108}{10v^2} - \frac{3}{10} &= m.\end{aligned}$$

From the first equation,

$$\begin{aligned}9 &= -1.5v + 2mv \\ \Rightarrow \frac{3}{2}v + 9 &= 2mv \\ \Rightarrow \frac{3}{4} + \frac{9}{2v} &= m \\ \Rightarrow \frac{3}{4} + \frac{9}{2v} &= \frac{108}{10v^2} - \frac{3}{10} \quad (\times 20v^2) \\ \Rightarrow 15v^2 + 90v &= 216 - 6v^2 \\ \Rightarrow 21v^2 + 90v - 216 &= 0 \\ \Rightarrow 7v^2 + 30v - 72 &= 0 \\ \Rightarrow v &= \frac{12}{7}, \not{-6}.\end{aligned}$$

(ii) From earlier work,

$$\begin{aligned}e &= \frac{v}{2} \\ &= \frac{6}{7}.\end{aligned}$$

(b) (i) Let the speed of P before the collision be u. As the speed of Q after the collision is v we have the following information about the collision.

Before	Mass	After
$u_1 = u\cos\alpha\,\vec{i} + u\sin\alpha\,\vec{j}$	$m_1 = 3m$	$v_1 = p\,\vec{i} + u\sin\alpha\,\vec{j}$
$u_2 = 0$	$m_2 = 7m$	$v_2 = v\,\vec{i}$

We want u in terms of v and α. Applying our usual two equations of motion,

$$\begin{aligned}m_1 u_1 + m_2 u_2 &= m_1 v_1 + m_2 v_2 \\ \Rightarrow 3m(u\cos\alpha) + 7m(0) &= 3mp + 7mv \\ \Rightarrow 3u\cos\alpha &= 3p + 7v.\end{aligned}$$

$$
\begin{aligned}
v_1 - v_2 &= -e(u_1 - u_2) \\
\Rightarrow p - v &= -\frac{2}{7}(u\cos\alpha - 0) \\
\Rightarrow p &= v - \frac{2}{7}u\cos\alpha.
\end{aligned}
$$

Plugging this back into the first equation,

$$
\begin{aligned}
3u\cos\alpha &= 3\left(v - \frac{2}{7}u\cos\alpha\right) + 7v \\
&= 3v - \frac{6}{7}u\cos\alpha + 7v \\
\Rightarrow \frac{27}{7}u\cos\alpha &= 10v \\
\Rightarrow u &= \frac{70v}{27\cos\alpha}.
\end{aligned}
$$

(ii) If $\alpha = 30^\circ$ then retracing our steps,

$$
\begin{aligned}
u &= \frac{70v}{27\cos 30^\circ} \\
&= \frac{140v}{27\sqrt{3}}.
\end{aligned}
$$

Then

$$
\begin{aligned}
p &= v - \frac{2}{7}u\cos 30^\circ \\
&= v - \frac{2}{7}\frac{140v}{27\sqrt{3}}\frac{\sqrt{3}}{2} \\
&= v - \frac{1}{7}\frac{140v}{27} \\
&= \frac{7v}{27}.
\end{aligned}
$$

Therefore if θ is the angle of P after collision,

$$
\begin{aligned}
\tan\theta &= \frac{u\sin\alpha}{p} \\
&= \frac{\frac{140v}{27\sqrt{3}}\frac{1}{2}}{\frac{7v}{27}} \\
&= \frac{\frac{70}{\sqrt{3}}}{7} \\
&= \frac{10}{\sqrt{3}} \\
\Rightarrow \theta &= 80^\circ
\end{aligned}
$$

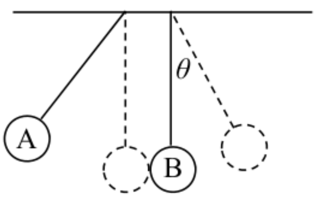
so that P is deflected

$$
80^\circ - 30^\circ = 50^\circ
$$

by the collision.

Question — 2016 Q5.

(a) Two small smooth spheres A, of mass 2 kg, and B, of mass 3 kg, are suspended by light strings from a ceiling as show in the diagram. The distance from the ceiling to the centre of each sphere is 2 m.



Sphere A is drawn back 60° and released from rest. A collides with B and rebounds. B swings through an angle θ.

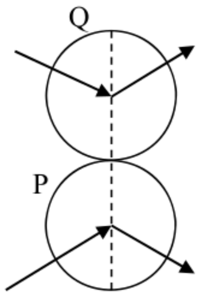
The coefficient of restitution between the spheres is $\frac{3}{4}$.

(i) Show that A strikes B with a speed of $\sqrt{2g}$ m s^{-1}.

(ii) Find the speed of each sphere after the collision.

(iii) Find the value of θ.

(b) Two identical smooth spheres P and Q collide.



The velocity of P **after** impact is $3\vec{i} - \vec{j}$ and the velocity of Q **after** impact is $2\vec{i} + \vec{j}$, where $\vec{j}$ is along the line of the centres of the spheres at impact.

The coefficient of restitution between the spheres is $\frac{1}{2}$.

Find

(i) the velocities, in terms of $\vec{i}$ and $\vec{j}$, of the two spheres before impact

(ii) to the nearest degree, the angle through which the direction of motion of P is deflected by the collision.

(a) (i) When calculating potential energies we will consider the initial position of A and B to be height 0. If A is swung back 60° then its height h can easily be calculated.

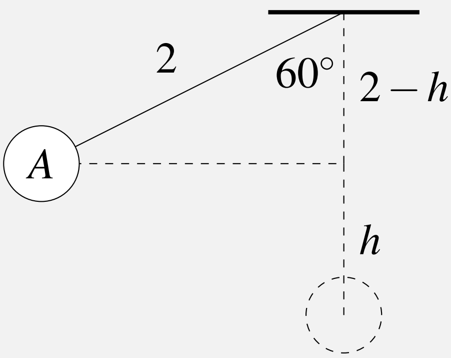


Figure 1

$$\cos 60^\circ = \frac{2-h}{2}$$
$$\Rightarrow 1 = 2-h$$
$$\Rightarrow 1 = h.$$

If v is its velocity just before it hits B then

$$\text{P.E.}_1 + \text{K.E.}_1 = \text{P.E.}_2 + \text{K.E.}_2$$
$$\Rightarrow mg(1) + 0 = 0 + \frac{mv^2}{2}$$
$$\Rightarrow 2g = v^2$$
$$\Rightarrow \sqrt{2g} = v.$$

(ii)

$$m_1u_1 + m_2u_2 = m_1v_1 + m_2v_2$$
$$\Rightarrow 2\sqrt{2g} + 0 = 2v_1 + 3v_2$$
$$\Rightarrow 2\sqrt{2g} = 2v_1 + 3v_2.$$

$$v_1 - v_2 = -e(u_1 - u_2)$$
$$= -\frac{3}{4}\sqrt{2g}.$$

Solving this system of equations by substitution,

$$v_1 - v_2 = -e\sqrt{2g}$$
$$v_1 = v_2 - \frac{3}{4}\sqrt{2g}.$$

Then

$$2\sqrt{2g} = 2v_1 + 3v_2$$
$$\Rightarrow 2\sqrt{2g} = 2\left(v_2 - \frac{3}{4}\sqrt{2g}\right) + 3v_2$$
$$= 5v_2 - \frac{3}{2}\sqrt{2g}$$
$$\Rightarrow \frac{7}{2}\sqrt{2g} = 5v_2$$
$$\Rightarrow \frac{7}{10}\sqrt{2g} = v_2.$$

Then

$$v_1 = v_2 - \frac{3}{4}\sqrt{2g}$$
$$= \frac{7}{10}\sqrt{2g} - \frac{3}{4}\sqrt{2g}$$
$$= -\frac{1}{20}\sqrt{2g}.$$

(iii) First lets find the height that B rises to. If it rises to a height of h before coming to rest then

$$\begin{aligned} \text{P.E.}_1 + \text{K.E.}_1 &= \text{P.E.}_2 + \text{K.E.}_2 \\ \Rightarrow 0 + \frac{m\left(\frac{7}{10}\sqrt{2g}\right)^2}{2} &= mgh + 0 \\ \Rightarrow \frac{49g}{100} &= gh \\ \Rightarrow 0.49 &= h. \end{aligned}$$

Then our diagram is as follows.

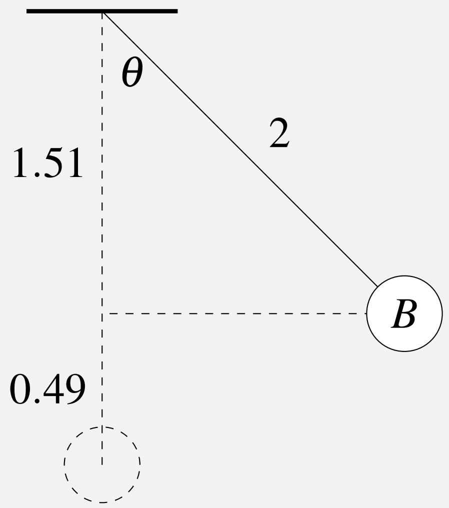


Figure 2

Then

$$\begin{aligned} \cos\theta &= \frac{1.51}{2} \\ \Rightarrow \theta &\approx 41^\circ. \end{aligned}$$

(b) (i) This time, in contrast to most exam questions, the $\vec{i}$ velocities are the ones that don't change. Also the $\vec{j}$ velocities after are known, but not before. So letting the $\vec{j}$ velocities before collision be p and q this is the information we know about the collision.

Before	Mass	After
$u_1 = 3\vec{i} + p\vec{j}$	$m_1 = m$	$v_1 = 3\vec{i} - \vec{j}$
$u_2 = 2\vec{i} + q\vec{j}$	$m_2 = m$	$v_2 = 2\vec{i} + \vec{j}$

Applying our usual two equations to the $\vec{j}$ components of the velocities,

$$\begin{aligned} m_1u_1 + m_2u_2 &= m_1v_1 + m_2v_2 \\ \Rightarrow mp + mq &= m(-1) + m(1) \\ \Rightarrow p + q &= 0. \end{aligned}$$

$$
\begin{aligned}
v_1 - v_2 &= -e(u_1 - u_2) \\
\Rightarrow -1 - 1 &= -\frac{1}{2}(p - q) \\
\Rightarrow 4 &= p - q.
\end{aligned}
$$

Solving

$$
\begin{aligned}
p + q &= 0, \\
p - q &= 4
\end{aligned}
$$

gives

$$
\begin{aligned}
p &= 2, \\
q &= -2.
\end{aligned}
$$

(ii) The angle of inclination of the velocity of P before collision is

$$\tan^{-1}\left(\frac{2}{3}\right) = 33.69^\circ.$$

The angle of inclination of the velocity of P after collision is

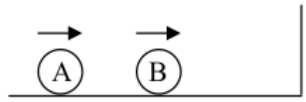
$$\tan^{-1}\left(\frac{-1}{3}\right) = -18.43^\circ.$$

Therefore the angle of deflection is

$$33.69^\circ - (-18.43^\circ) \approx 52^\circ.$$

Question — 2015 Q5.

(a) A small smooth sphere A, of mass $2m$, moving with speed $9u$ m s^{-1}, collides directly with a small smooth sphere B, of mass $5m$, which is moving in the same direction with speed $2u$ m s^{-1}.



Sphere B then collides with a vertical wall, rebounds and collides again with sphere A. The wall is perpendicular to the direction of motion of the spheres.

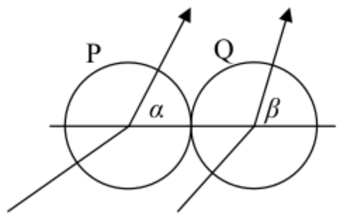
The first collision takes place 35 cm from the wall.

The coefficient of restitution between the spheres is $\frac{4}{5}$.

The coefficient of restitution between sphere B and the wall is $\frac{5}{14}$.

(i) Show that, as a result of the first collision, A comes to rest.
(ii) Find the time between the two collisions between A and B in terms of u.

(b) Two identical smooth spheres, P and Q, collide.



The coefficient of restitution is 1.

The velocity of P before impact is $a\vec{i} + b\vec{j}$ and the velocity of Q before impact is $c\vec{i} + d\vec{j}$, where $\vec{i}$ is along the line of the centres of the spheres at impact.

After impact the direction of motion of P makes an angle α with their line of centres and the direction of motion of Q makes an angle β with their line of centres.

Show that $\tan\alpha \tan\beta = \frac{bd}{ac}$.

(a) (i) First studying the collision between A and B,

$$\begin{aligned} m_1u_1 + m_2u_2 &= m_1v_1 + m_2v_2 \\ \Rightarrow 2m(9u) + 5m(2u) &= 2mv_1 + 5mv_2 \\ \Rightarrow 28u &= 2v_1 + 5v_2. \end{aligned}$$

$$\begin{aligned} v_1 - v_2 &= -e(u_1 - u_2) \\ &= -\frac{4}{5}(9u - 2u) \\ &= -\frac{28u}{5} \\ \Rightarrow 5v_1 - 5v_2 &= -28u. \end{aligned}$$

Adding these equations,

$$\begin{aligned} 2v_1 + 5v_2 &= 28u \\ (+)\ 5v_1 - 5v_2 &= -28u \\ \hline 7v_1 \qquad &= 0 \\ \Rightarrow v_1 &= 0. \end{aligned}$$

(ii) If $v_1 = 0$ then

$$\begin{aligned} 28u &= 0 + 5v_2 \\ \Rightarrow \frac{28u}{5} &= v_2. \end{aligned}$$

As this is turning into a linear motion problem it's simpler to find the **speed** of B after it hits the wall. If this speed is v_3 then

$$\begin{aligned} v_3 &= \frac{5}{14}v_2 \\ &= \frac{5}{14}\left(\frac{28u}{5}\right) \\ &= 2u. \end{aligned}$$

The total time taken for B to complete both 35 cm trips from A's resting point to the wall is

$$\frac{0.35}{\frac{28u}{5}} + \frac{0.35}{2u} = \frac{19}{80u}.$$

(b) As the spheres are identical we can assume they are both of mass m.

Before	Mass	After
$u_1 = a\,\vec{i} + b\,\vec{j}$	$m_1 = m$	$v_1 = p\,\vec{i} + b\,\vec{j}$
$u_2 = c\,\vec{i} + d\,\vec{j}$	$m_2 = m$	$v_2 = q\,\vec{i} + d\,\vec{j}$

Applying our usual two equations of motion,

$$\begin{aligned} m_1u_1 + m_2u_2 &= m_1v_1 + m_2v_2 \\ \Rightarrow ma + mc &= mp + mq \\ \Rightarrow a + c &= p + q. \end{aligned}$$

$$\begin{aligned} v_1 - v_2 &= -e(u_1 - u_2) \\ \Rightarrow p - q &= -(a - c) \\ &= b - a. \end{aligned}$$

Adding these equations,

$$\begin{aligned} p+q &= a+c \\ (+)\ p-q &= c-a \\ \hline 2p &= 2c \\ \Rightarrow p &= c. \end{aligned}$$

Then

$$\begin{aligned} p+q &= a+c \\ \Rightarrow c+q &= a+c \\ \Rightarrow q &= a. \end{aligned}$$

Therefore

$$\begin{aligned} \tan\alpha\tan\beta &= \frac{b}{p}\left(\frac{d}{q}\right) \\ &= \frac{b}{c}\left(\frac{d}{a}\right) \\ &= \frac{bd}{ac}. \end{aligned}$$