Circular Motion Exam Question Solutions

Note These exam questions are given in reverse chronological order as they appear in exam papers; 2023 paper, Sample paper, 2022 (deferred), 2022, and so on back to 2015. For the sake of including questions on horizontal circular motion due to friction exam questions from 2006 and 2000 are also included. All questions are answered in the style described in my notes. Only questions from the old syllabus relevant to the new syllabus are included.

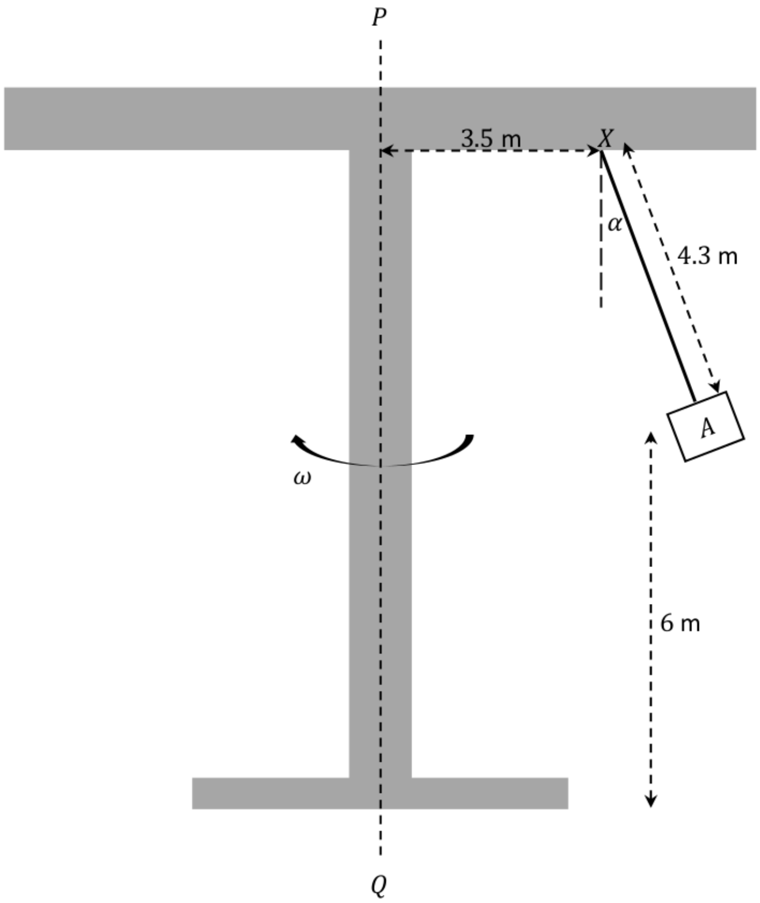
Question — 2023 Q3.

Question 3

The photograph on the right is of a chain swing ride in an amusement park. The disk at the top of the ride is rotating in a horizontal plane. People sit in seats which are attached freely by inextensible chains of length 4.3 m to fixed points on the disk.

The chain attaching seat A hangs from point X on the ride and makes an angle α with the vertical. X is 3.5 m from the axis of rotation, which is the vertical line PQ, as shown in the diagram below. The chain is free to swing in or out relative to PQ.

The ride rotates about PQ with constant angular velocity ω. Seat A moves in a horizontal circular path which is 6 m above the ground.



(i) Draw a diagram to show the external forces acting on seat A.

(ii) Show that $\omega = \sqrt{\frac{g \tan \alpha}{3.5+4.3 \sin \alpha}}$.

(iii) Use dimensional analysis to show that the units for the expression $\sqrt{\frac{g \tan \alpha}{3.5+4.3 \sin \alpha}}$ are equivalent to the units for ω.

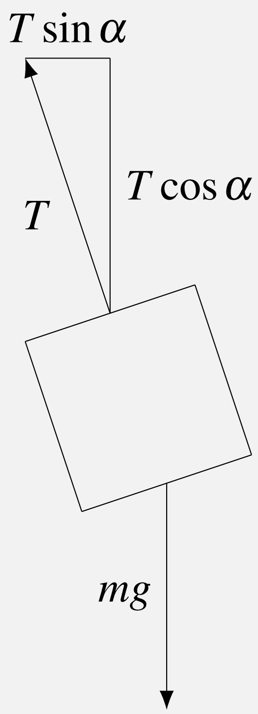
It is found by measurement that $\alpha = 25°$.

(iv) Calculate how many complete revolutions the ride makes in one minute.

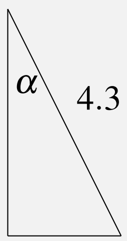
The person sitting in seat A throws a small orange into the air. The person imparts an upward vertical velocity component of 4 m s^{-1} to the orange.

(v) Calculate the time from when the orange is thrown until it hits the ground.

(i) Note that the resolving of the T force is not necessary for full marks, but will be used later.



(ii) We will use ω and r in later calculations, as ω is in the answer and we can find r (at least in terms of α). Considering the right-angled triangle with the chain as the hypotenuse,



we can see that the horizontal line has length $4.3\sin\alpha$ and so the radius of the circle that the seat describes is equal to

$$r = 3.5 + 4.3\sin\alpha.$$

Letting our upwards forces equal our downwards forces we get

$$\begin{aligned} T\cos\alpha &= mg \\ \Rightarrow T &= \frac{mg}{\cos\alpha}. \end{aligned}$$

Our centripetal force equation is then

$$
\begin{aligned}
F &= m\omega^2 r \\
\Rightarrow T\sin\alpha &= m\omega^2(3.5+4.3\sin\alpha) \\
\Rightarrow \frac{mg\sin\alpha}{\cos\alpha} &= m\omega^2(3.5+4.3\sin\alpha) \\
\Rightarrow mg\tan\alpha &= m\omega^2(3.5+4.3\sin\alpha) \\
\Rightarrow \frac{mg\tan\alpha}{3.5+4.3\sin\alpha} &= \omega^2 \\
\Rightarrow \sqrt{\frac{mg\tan\alpha}{3.5+4.3\sin\alpha}} &= \omega.
\end{aligned}
$$

(iii) (Note: I don't like this question, and believe it to be badly written. For this question to make sense 3.5 and 4.3 should have been replaced with variables like d and l so that they could be easier understood as variables given in metres rather than just numbers).

3.5 and 4.3 are measured in metres (m) and the trigonometric ratios $\sin\alpha$ and $\cos\alpha$ are unit-free. g is measured in m/s^2 and so

$$
\begin{aligned}
\omega &= \sqrt{\frac{\frac{m}{s^2}}{\mathrm{m}+m}} \\
&= \sqrt{\frac{\frac{m}{s^2}}{m}} \\
&= \sqrt{\frac{1}{s^2}} \\
&= \frac{1}{s} \\
&= \text{rad/sec}
\end{aligned}
$$

as radians are "unit-free" in the SI sense.

(iv) If $\alpha = 25^\circ$ then

$$
\begin{aligned}
\omega &= \sqrt{\frac{mg\tan 25^\circ}{3.5+4.3\sin 25^\circ}} \\
&= 0.927
\end{aligned}
$$

and the number of revolutions it makes per minute is 60 times the number of revolutions per second:

$$60 \times \frac{\omega}{2\pi} = 8.85$$

so that it makes 8 complete revolutions per minute. (Note: full marks were given for an answer of 8 or 9).

(v) The sideways motion is irrelevant to the question, and this can be treated as a freefall problem:

$$
\begin{aligned}
u &= 4 \\
v &= \\
a &= -g \\
s &= -6 \\
t &=
\end{aligned}
$$

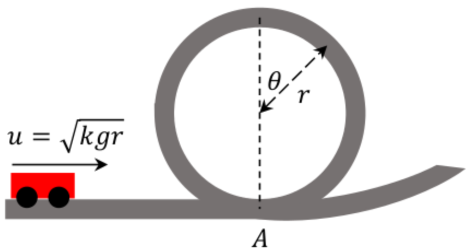
$$
\begin{aligned}
s &= ut + \frac{1}{2}at^2 \\
\Rightarrow -6 &= 4t - 4.9t^2 \\
\Rightarrow 4.9t^2 - 4t - 6 &= 0 \\
\Rightarrow t &= 1.59 \text{ seconds.}
\end{aligned}
$$

Question — 2023 Q10 (b).

(b) A toy car track consists of a series of components that connect to make a closed circuit. Part of the track makes a vertical circular loop.

To model the motion of a car on this track, its velocity at the base of the loop (point A) is expressed as $u = \sqrt{kgr}$, where r is the radius of the loop, g is the acceleration due to gravity, and k is a constant.

The model ignores the effects of friction.

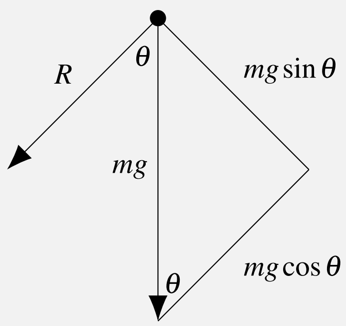


(i) Draw a diagram to show the forces acting on the car at the instant when the radius to the car makes an angle θ with the upward vertical.

(ii) If the car loses contact with the track at the instant when the radius to the car makes an angle θ with the upward vertical, show that $\cos\theta = \frac{k-2}{3}$.

(iii) Calculate the minimum value of k such that the car successfully completes the loop without losing contact with the track.

(i) Note that the resolving of the mg force is unnecessary for full marks, but will be used later.



(ii) At the moment the car loses contact with the track $R = 0$. Assuming a speed of v at this time the centripetal force equation gives us

$$\begin{aligned} F &= m\frac{v^2}{r} \\ \Rightarrow mg\cos\theta &= m\frac{v^2}{r} \\ \Rightarrow gr\cos\theta &= v^2. \end{aligned}$$

Regarding the Principle of Conservation of Energy, if we measure height from point A then the height of the object when the object loses contact with the track is

$$r + r\cos\theta.$$

Then using the Principle of Conservation of Energy to compare the energies at this point with that at point A,

$$\begin{aligned}
\text{P.E.}_1 + \text{K.E.}_1 &= \text{P.E.}_2 + \text{K.E.}_2 \\
\Rightarrow 0 + \frac{m(\sqrt{kgr})^2}{2} &= mg\,(r + r\cos\theta) + \frac{m(gr\cos\theta)}{2} \\
\Rightarrow mkgr &= 2mgr + 2mgr\cos\theta + mgr\cos\theta \\
\Rightarrow k &= 2 + 3\cos\theta \\
\Rightarrow \frac{k-2}{3} &= \cos\theta.
\end{aligned}$$

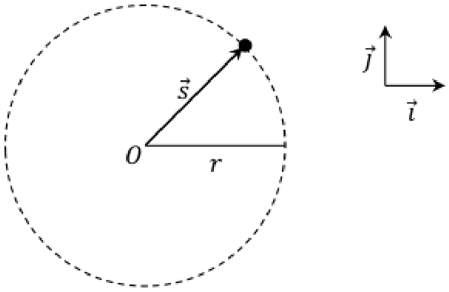
(iii) We usually say that $R = 0$ at the top of the circle (when $\theta = 0$) to find the minimum value of k such that the car moves in a complete circle, so using the work from (ii) by simply setting $\theta = 0$,

$$\begin{aligned}
\frac{k-2}{3} &= \cos 0^\circ \\
\Rightarrow k &= 5.
\end{aligned}$$

Question — Sample Q6.

Question 6

A learner driver is practising driving around a roundabout.



The motion of the car may be modelled as horizontal circular motion around centre O, with radius r and constant angular speed ω, as in the diagram above.

(i) Write an expression for $\vec{s}$, the displacement of the car relative to O at any time t, in terms of r, ω and t. Your expression should use the unit vectors $\vec{\imath}$ and $\vec{\jmath}$.

Note that $t = 0$ when $\vec{s}$ is along the $\vec{\imath}$ axis.

(ii) Derive an expression for $\vec{v}$, the velocity of the car at any time t.

(iii) Use a dot product calculation to show that the car's velocity and displacement are always perpendicular to each other.

(iv) Show that the acceleration of the car is always directed towards O.

(v) Derive an expression for the maximum velocity the car could have as it travels around the roundabout, without slipping. Your expression should be written in terms of r, g and μ, the coefficient of friction between the car and the road.

(vi) Use dimensional analysis to show that the units for the expression you derived in part **(v)** are equivalent to the units for velocity.

(vii) Do you think the assumptions made in developing this model were appropriate? Explain your answer.

(i)

$$\vec{s} = r\cos(\omega t)\,\vec{\imath} + r\sin(\omega t)\,\vec{\jmath}$$

(ii)

$$\begin{aligned}\vec{v} &= \frac{d\vec{s}}{dt} \\ &= -\omega r\sin(\omega t)\,\vec{\imath} + \omega r\cos(\omega t)\,\vec{\jmath}\end{aligned}$$

(iii)

$$\begin{aligned}\vec{v}\cdot\vec{s} &= -\omega r\sin(\omega t)\,[r\cos(\omega t)] + \omega r\cos(\omega t)\,[r\sin(\omega t)]\\ &= -\omega r^2\cos(\omega t)\sin(\omega t) + \omega r^2\cos(\omega t)\sin(\omega t)\\ &= 0.\end{aligned}$$

(iv)

$$\begin{aligned}\vec{a} &= \frac{d\vec{v}}{dt}\\ &= -\omega^2 r\cos(\omega t)\,\vec{i} - \omega^2 r\sin(\omega t)\,\vec{j}\\ &= -\omega^2\left(r\cos(\omega t)\,\vec{i} + r\sin(\omega t)\,\vec{j}\right)\\ &= -\omega^2\vec{s}.\end{aligned}$$

As $\vec{a}$ is a negative multiple of $\vec{s}$ it points towards the centre of the circle.

(v) We assume that the car is moving in a circle but that the limiting friction is being applied to the car to find the maximum speed v that causes the car not to skid. The mass is unknown and so is given by m. The following forces are acting on the car, where this view is from behind the car as it turns to the left.

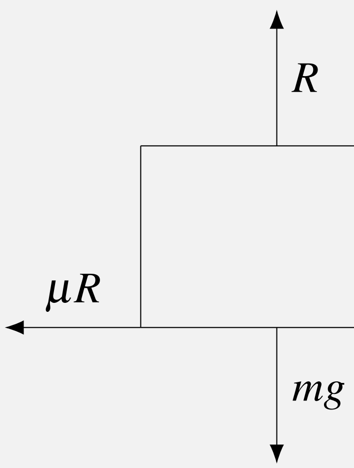


Figure 1

Therefore $R = mg$ and so $\mu R = \mu mg$ is equal to the centripetal force. Then

$$\begin{aligned}F &= m\frac{v^2}{r}\\ \Rightarrow \mu mg &= m\frac{v^2}{r}\\ \Rightarrow \mu gr &= v^2\\ \Rightarrow \sqrt{\mu gr} &= v.\end{aligned}$$

(vi) μ is unit free, g is given in m/s^2 and r is given in metres, so that

$$\begin{aligned} v &= \sqrt{\frac{m}{s^2}m} \\ &= \sqrt{\frac{m^2}{s^2}} \\ &= \frac{m}{s} \end{aligned}$$

which are the units for v.

(vii) There are so many correct answers to a question like this, but most reference an over simplifying assumption and state whether the oversimplification makes the model behave sizeably differently to the real life situation. For example, “The car is probably not travelling in an exact circle, but the effects of this are minimal in real life and assuming the car travels in an exact circle is a reasonable model” would be one answer. “The car is probably not travelling at uniform angular speed for its entire time on the roundabout, and may in fact not travel at a constant speed for any time period, making this model inappropriate for the car” would be another.

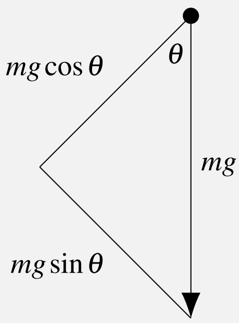
Question — 2022 (Deferred) Q6 (b).

(b) A particle of mass m is suspended vertically from a fixed point O by a light inelastic string of length d metres.

The particle is projected horizontally with speed u, where $u^2 = 4gd$.

Show the string goes slack when it makes an angle $\cos^{-1}\frac{2}{3}$ with the upward vertical through O.

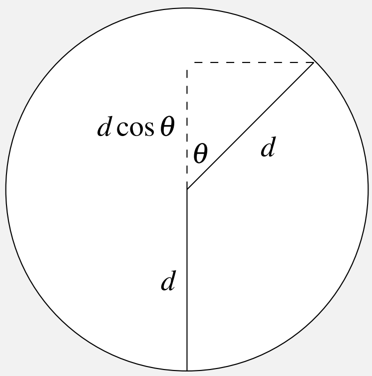
If the string goes slack when it makes an angle of θ with the upward vertical, then $T = 0$ at this point and only force acting on the particle is gravity, which looks as follows when resolved into and tangent to the circle.



Assuming a speed of v at this time, the centripetal force equation gives us

$$F = m\frac{v^2}{r}$$
$$\Rightarrow mg\cos\theta = m\frac{v^2}{d}$$
$$\Rightarrow dg\cos\theta = v^2.$$

Regarding the Principle of Consrvation of Energy, if we measure height from the bottom of the circle then when the string goes slack it is at a height of $d + d\cos\theta$.



Therefore comparing the energy this point with that of the point at the bottom of the circle,

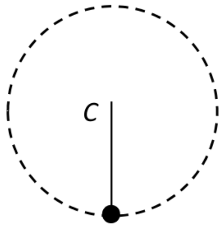
$$
\begin{aligned}
\text{P.E.}_1 + \text{K.E.}_1 &= \text{P.E.}_2 + \text{K.E.}_2 \\
\Rightarrow 0 + \frac{m(4gd)}{2} &= mg\,(d + d\cos\theta) + \frac{m(dg\cos\theta)}{2} \\
\Rightarrow 4mgd &= 2mgd + 2mgd\cos\theta + mgd\cos\theta \\
\Rightarrow 2mgd &= 3mgd\cos\theta \\
\Rightarrow \frac{2}{3} &= \cos\theta \\
\Rightarrow \cos^{-1}\left(\frac{2}{3}\right) &= \theta.
\end{aligned}
$$

Question — 2022 Q6 (b).

(b) A particle is attached to one end of a light inextensible string of length 0.5 m. The other end of the string is attached to a fixed point C. The particle moves in a vertical circle.

The greatest and least tensions in the string are $3T$ and T, respectively.

Find the speed of the particle at the lowest point.



The greatest and least tension happen at the bottom and top of the circle respectively. Let the speeds at these points be u and v.

The forces acting on the particle at the top of the circle are as shown (giving the particle a mass of m),

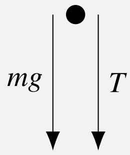


Figure 2

Then the centripetal force equation is

$$\begin{aligned} F &= m\frac{v^2}{r} \\ \Rightarrow T + mg &= m\frac{v^2}{0.5} \\ \Rightarrow T + mg &= 2mv^2. \end{aligned}$$

The forces acting on the particle at the bottom of the circle are as shown.

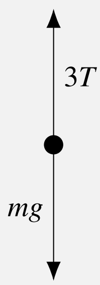


Figure 3

Then the centripetal force equation is

$$\begin{aligned} F &= m\frac{v^2}{r} \\ \Rightarrow 3T - mg &= m\frac{u^2}{0.5} \\ \Rightarrow 3T - mg &= 2mu^2. \end{aligned}$$

Using the first equation to remove T from this equation,

$$\begin{aligned} 3T - mg &= 2mu^2 \\ \Rightarrow 3(2mv^2 - mg) - mg &= 2mu^2 \\ \Rightarrow 6mv^2 - 3mg - mg &= 2mu^2 \\ \Rightarrow 3v^2 - 2g &= u^2. \end{aligned}$$

Using the Principle of Conservation of Energy to compare these two points, and measuring height from the bottom of the sphere,

$$\begin{aligned} \text{P.E.}_1 + \text{K.E.}_1 &= \text{P.E.}_2 + \text{K.E.}_2 \\ \Rightarrow 0 + \frac{mu^2}{2} &= mg(1) + \frac{mv^2}{2} \\ \Rightarrow u^2 &= 2g + v^2 \\ \Rightarrow u^2 - 2g &= v^2. \end{aligned}$$

Using this equation to substitute v^2 out of the second equation,

$$\begin{aligned} 3v^2 - 2g &= u^2 \\ \Rightarrow 3(u^2 - 2g) - 2g &= u^2 \\ \Rightarrow 3u^2 - 6g - 2g &= u^2 \\ \Rightarrow 2u^2 &= 8g \\ \Rightarrow u &= \sqrt{4g}. \end{aligned}$$

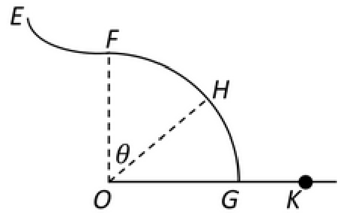
Question — 2021 Q6 (b).

(b) A smooth slide *EFG* is in the shape of two arcs, *EF* and *FG*, each of radius *r*. The centre *O* of arc *FG* is vertically below *F* as shown in the diagram.

Point *E* is at a height $\frac{r}{5}$ above point *F*.

A child starts from rest at *E*, moves along the slide past the point *F* and loses contact with the slide at point *H*.

OH makes an angle θ with the vertical.

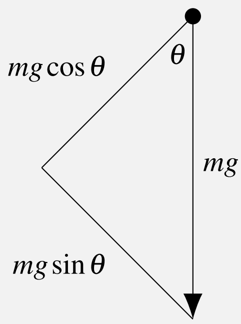


(i) Find the value of θ.

The child lands in a sandpit at point *K*.

(ii) Find, in terms of *r*, the speed of the child at *K*.

(i) At the moment the particle loses contact with the slide the reaction force is equal to 0 and the only force acting on the particle is gravity, which looks as follows when resolved into and tangent to the circle.



Then assuming a speed of v at this point the centripetal force equation gives us

$$F = m\frac{v^2}{r}$$
$$\Rightarrow mg\cos\theta = m\frac{v^2}{r}$$
$$\Rightarrow gr\cos\theta = v^2.$$

Using the Principle of Conservation of Energy to compare the energies at points E and H, measuring height from O and noting that H is $r\cos\theta$ metres above O,

$$\text{P.E.}_1 + \text{K.E.}_1 = \text{P.E.}_2 + \text{K.E.}_2$$
$$\Rightarrow mg\left(\frac{6r}{5}\right) + 0 = mg(r\cos\theta) + \frac{mv^2}{2}$$
$$12mgr = 10mgr\cos\theta + 5m(gr\cos\theta)$$
$$\Rightarrow 12mgr = 15mgr\cos\theta$$
$$\Rightarrow \frac{4}{5} = \cos\theta$$
$$\Rightarrow 37^\circ = \theta.$$

(ii) We could treat this as a projectiles problem, but since we only care about the speed of impact we could simply apply the Principle of Conservation of Energy to points H and K. Note that if $\cos\theta = \frac{4}{5}$ that H is $4r$ metres above K. Also note that the speed of the particle at H satisfies

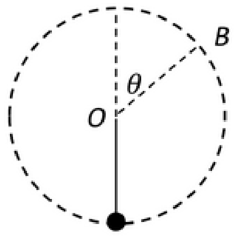
$$\begin{aligned} v^2 &= gr\cos\theta \\ &= \frac{4gr}{5} \end{aligned}$$

Then if we give the particle a speed of w at K,

$$\begin{aligned} \text{P.E.}_1 + \text{K.E.}_1 &= \text{P.E.}_2 + \text{K.E.}_2 \\ \Rightarrow mg\left(\frac{4r}{5}\right) + \frac{m\frac{4gr}{5}}{2} &= 0 + \frac{mw^2}{2} \\ \frac{4mgr}{5} + \frac{2mgr}{5} &= \frac{mw^2}{2} \\ \Rightarrow \frac{12mgr}{5} &= mw^2 \\ \Rightarrow \sqrt{\frac{12g}{5}r} &= w. \end{aligned}$$

Question — 2020 Q6 (b).

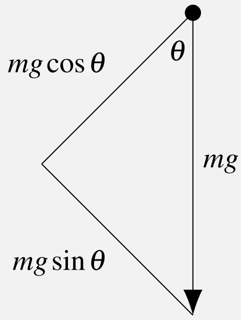
(b) A particle P is attached to one end of a light inextensible string of length d.
The other end of the string is attached to a fixed point O.
The particle is hanging freely at rest, with the string vertical, when it is projected horizontally with speed $\sqrt{3gd}$.
The particle moves in a vertical circle.
The string becomes slack when P is at the point B.
OB makes an angle θ with the upward vertical.



(i) Show that $\cos\theta = \frac{1}{3}$.

(ii) In terms of d, find the greatest height of P above B in the subsequent motion.

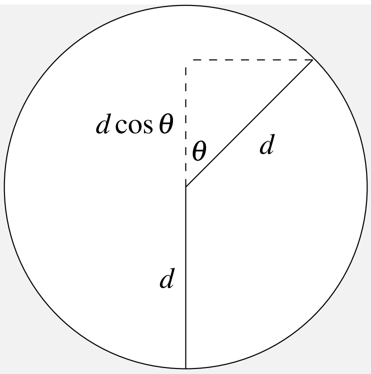
(i) If the string goes slack when it makes an angle of θ with the upward vertical, then $T = 0$ at this point and only force acting on the particle is gravity, which looks as follows when resolved into and tangent to the circle.



Assuming a speed of v at this time, the centripetal force equation gives us

$$F = m\frac{v^2}{r}$$
$$\Rightarrow mg\cos\theta = m\frac{v^2}{d}$$
$$\Rightarrow dg\cos\theta = v^2.$$

Regarding the Principle of Consrvation of Energy, if we measure height from the bottom of the circle then when the string goes slack it is at a height of $d + d\cos\theta$.



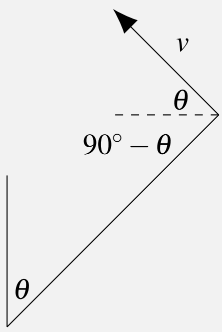
Therefore comparing the energy this point with that of the point at the bottom of the circle,

$$
\begin{aligned}
\text{P.E.}_1 + \text{K.E.}_1 &= \text{P.E.}_2 + \text{K.E.}_2 \\
\Rightarrow 0 + \frac{m\sqrt{3gd}^2}{2} &= mg\,(d + d\cos\theta) + \frac{m(dg\cos\theta)}{2} \\
\Rightarrow 3mgd &= 2mgd + 2mgd\cos\theta + mgd\cos\theta \\
\Rightarrow mgd &= 3mgd\cos\theta \\
\Rightarrow \frac{1}{3} &= \cos\theta.
\end{aligned}
$$

(ii) This is a projectiles problem, where the initial speed of the object is

$$
\begin{aligned}
v &= \sqrt{dg\cos\theta} \\
&= \sqrt{\frac{dg}{3}}
\end{aligned}
$$

and its initial projection angle is θ:



Drawing a triangle to show that $\sin\theta = \frac{\sqrt{8}}{3}$ using Pythagoras' Theorem, we have the following UVAST array for the projectiles problem (although we only need the y column):

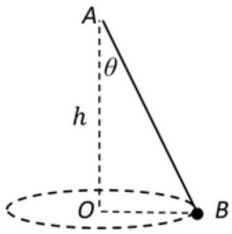
$$
\begin{array}{lll}
\underline{x\text{-axis}} & \underline{y\text{-axis}} & \\
u_x = \sqrt{\frac{dg}{3}}\frac{1}{3} & u_y = \sqrt{\frac{dg}{3}}\frac{1}{\sqrt{10}} & \\
v_x = \sqrt{\frac{dg}{3}}\frac{1}{3} & u_y = \sqrt{\frac{dg}{3}}\frac{1}{\sqrt{10}} & v_y = 0 \\
a_x = 0 & a_y = -g & \\
s_x = & s_y = & \\
t_x = t & t_y = t &
\end{array}
$$

Then

$$\begin{aligned}
v_y^2 &= u_y^2 + 2a_y s_y \\
\Rightarrow 0 &= \frac{dg}{3}\frac{8}{9} - 2gs_y \\
\Rightarrow 2s_y &= \frac{8d}{27} \\
\Rightarrow s_y &= \frac{4d}{27}.
\end{aligned}$$

Question — 2019 Q6 (a).

6. **(a)** One end A of a light elastic string is attached to a fixed point. The other end, B, of the string is attached to a particle of mass m. The particle moves on a smooth horizontal table in a circle with centre O, where O is vertically below A and $|AO| = h$. The string makes an angle θ with the downward vertical and B moves with constant angular speed ω about OA.



(i) Show that $\omega^2 \leq \frac{g}{h}$.

The elastic string has natural length h and elastic constant $\frac{2mg}{h}$.

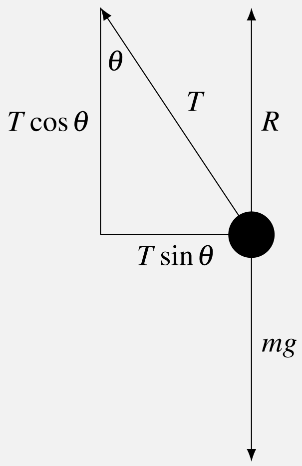
(ii) Given that $\omega^2 = \frac{2g}{5h}$, find the value of θ.

(i) First, r and ω will be the variables we use for our centripetal force equation later. We can get r in terms of h and θ:

$$\tan\theta = \frac{r}{h}$$
$$\Rightarrow h\tan\theta = r$$

so that r is at least in terms of variables given in the question.

The following diagram shows the forces acting on the particle.



As upwards forces equal downwards forces,

$$R + T\cos\theta = mg.$$

The centripetal force equation gives us

$$F = m\omega^2 r$$
$$\Rightarrow T\sin\theta = m\omega^2 h\tan\theta.$$

If ω is large enough the particle will lift off the table. Assuming the particle is on the point of lifting off the table, $R = 0$ and

$$\begin{aligned} 0 + T\cos\theta &= mg \\ \Rightarrow T &= \frac{mg}{\cos\theta}. \end{aligned}$$

Then

$$\begin{aligned} T\sin\theta &= m\omega^2 h\tan\theta \\ \Rightarrow \frac{mg}{\cos\theta}\sin\theta &= m\omega^2 h\tan\theta \\ \Rightarrow g\tan\theta &= \omega^2 h\tan\theta \\ \Rightarrow \frac{g}{h} &= \omega^2. \end{aligned}$$

Therefore is the particle is on the table then $\omega^2 \leq \frac{g}{h}$.

(ii) Our original two simultaneous equations are true, and if $\omega^2 = \frac{2g}{5h}$ then the second simplifies.

$$\begin{aligned} R + T\cos\theta &= mg, \\ T\sin\theta &= m\left(\frac{2g}{5h}\right)h\tan\theta \\ &= \frac{2mg}{5}\tan\theta. \end{aligned}$$

Our third equation is from Hooke's Law. Noting that the length of the string, l, can be written in terms of h and θ,

$$\begin{aligned} \cos\theta &= \frac{h}{l} \\ \Rightarrow l &= \frac{h}{\cos\theta}. \end{aligned}$$

Then

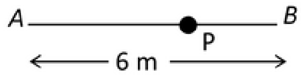
$$\begin{aligned} T &= k(l - l_0) \\ &= \frac{2mg}{h}\left(\frac{h}{\cos\theta} - h\right) \\ &= 2mg\left(\frac{1}{\cos\theta} - 1\right). \end{aligned}$$

Ignoring the first equation and putting this into the second,

$$
\begin{aligned}
2mg\left(\frac{1}{\cos\theta}-1\right)\sin\theta &= \frac{2mg}{5}\tan\theta \\
\Rightarrow \left(\frac{1}{\cos\theta}-1\right)\sin\theta &= \frac{1}{5}\frac{\sin\theta}{\cos\theta} \\
\Rightarrow \frac{1}{\cos\theta}-1 &= \frac{1}{5}\frac{1}{\cos\theta} \\
\Rightarrow \frac{4}{5}\frac{1}{\cos\theta} &= 1 \\
\Rightarrow \frac{4}{5} &= \cos\theta \\
\Rightarrow 37^\circ &= \theta.
\end{aligned}
$$

Question — 2018 Q6 (ignoring (a)(ii)).

(a) Two points A and B are 6 m apart on a smooth horizontal surface. A particle P of mass 0·5 kg is attached to one end of a light elastic string, of natural length 2·5 m and elastic constant 8 N m^{-1}. The other end of the string is attached to A.



A second light elastic string, of natural length 1·5 m and elastic constant 12 N m^{-1} has one end attached to P and the other end attached to B, as shown in the diagram.
Initially P rests in equilibrium at the point O, where AOB is a straight line.

(i) Find the length of AO.

Note: The mass can be ignored for part (i).

(b) A particle P is attached to one end of a light inextensible string of length d. The other end of the string is attached to a fixed point. The particle is hanging freely at rest, with the string vertical, when it is projected horizontally with speed u.
The particle moves in a complete vertical circle.

(i) Show that $u \geq \sqrt{5gd}$.

As P moves in the circle the least tension in the string is T_1 and the greatest tension is kT_1.

(ii) Given that $u = \sqrt{6gd}$, find the value of k.

(a) (i) Assume that the AO string is length x, so that the OB string is length $6-x$. Let S, T be the tension of the AO, OB string respectively. As the object is not moving and applying Hooke's Law,

$$\begin{aligned} S &= T \\ \Rightarrow 8(x-2.5) &= 12(6-x-1.5) \\ \Rightarrow 8x-20 &= 54-12x \\ \Rightarrow 20x &= 74 \\ \Rightarrow x &= 3.7 \text{ m.} \end{aligned}$$

(b) (i) Our points of interest are when the object is at the bottom and top of the circle. Assume the object has speed v at the top of the circle. Drawing the forces acting on the object at the top of the circle is quite straightforward.

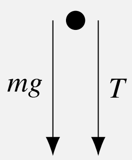


Figure 4

Therefore the centripetal force equation yields

$$\begin{aligned} F &= m\frac{mv^2}{r} \\ \Rightarrow T + mg &= \frac{mv^2}{d}. \end{aligned}$$

We could let $T \geq 0$ and deal with inequalities, but its easier to let $T = 0$. This assumes that the particle "just about" completes a circle, and so gives the minimum value of u such that the object travels in a complete circle. If $T = 0$,

$$\begin{aligned} T + mg &= \frac{mv^2}{d} \\ \Rightarrow mg &= \frac{mv^2}{d} \\ \Rightarrow dg &= v^2. \end{aligned}$$

Using the Principle of Conservation of Energy to compare the objects energy at the bottom and top of the circle, measuring potential energy from the bottom of the circle,

$$\begin{aligned} \text{P.E.}_1 + \text{K.E.}_1 &= \text{P.E.}_2 + \text{K.E.}_2 \\ \Rightarrow 0 + \frac{mu^2}{2} &= mg(2d) + \frac{mv^2}{2} \\ \Rightarrow u^2 &= 4dg + v^2 \\ &= 4dg + dg \\ &= 5dg \\ \Rightarrow u^2 &= 5dg \\ \Rightarrow u &= \sqrt{5dg}. \end{aligned}$$

As stated before, this is the minimum value of u such that the object completes a full circle. Therefore if the object completes a full circle $u \geq \sqrt{5dg}$.

(ii) The least and greatest tension occur when the particle is at the top and bottom of the circle respectively. Recreating the same two equations as in part (i) without assuming that $T = 0$ at the top of the circle, the centripetal force equation at the top of the circle is now

$$\begin{aligned} F &= m\frac{mv^2}{r} \\ \Rightarrow T_1 + mg &= \frac{mv^2}{d} \\ \Rightarrow dT_1 + dmg &= mv^2. \end{aligned}$$

The Principle of Conservation of Energy equation is now

$$
\begin{aligned}
\text{P.E.}_1 + \text{K.E.}_1 &= \text{P.E.}_2 + \text{K.E.}_2 \\
\Rightarrow 0 + \frac{m\sqrt{6gd}^2}{2} &= mg(2d) + \frac{mv^2}{2} \\
\Rightarrow 3dmg &= 2dmg + \frac{mv^2}{2} \\
\Rightarrow dmg &= \frac{mv^2}{2} \\
\Rightarrow 2dg &= v^2.
\end{aligned}
$$

Subistituting this into the first equation,

$$
\begin{aligned}
dT_1 + dmg &= mv^2 \\
\Rightarrow dT_1 + dmg &= 2dmg \\
\Rightarrow T_1 &= mg.
\end{aligned}
$$

The forces acting on the particle when it is at the bottom of the circle are as shown.

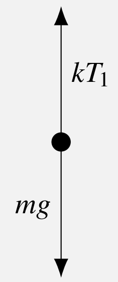


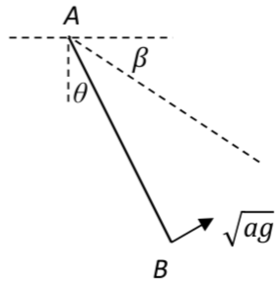
Figure 5

Then the centripetal force equation is

$$
\begin{aligned}
F &= m\frac{v^2}{r} \\
\Rightarrow kT_1 - mg &= m\frac{\sqrt{6gd}^2}{d} \\
\Rightarrow kmg - mg &= 6mg \\
\Rightarrow k &= 7.
\end{aligned}
$$

Question — 2017 Q6 (b).

(b) One end A of a light inextensible string of length $3a$ is attached to a fixed point. A particle of mass m is attached to the other end B of the string. The string makes an angle θ with the vertical.



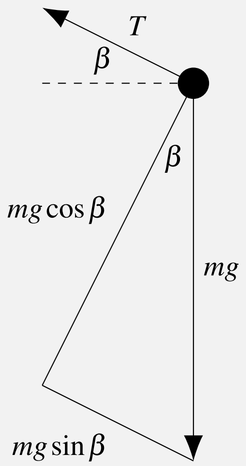
The particle is held in equilibrium with the string taut and $\cos\theta = \frac{2}{3}$. The particle is then projected with speed $\sqrt{ag}$, in the direction perpendicular to AB, as show in the diagram.
In the subsequent motion the string remains taut.

When AB makes an angle β below the horizontal, the speed of the particle is v and the tension in the string is T.

(i) Show that $v^2 = 3ag(2\sin\beta - 1)$.

(ii) Find the minimum value and the maximum value of T.

(i) When AB makes an angle of β with the horizontal, the diagram below shows the forces acting on the particle, resolved into and tangent to the circle.



Then the centripetal force equations yields

$$F = m\frac{v^2}{r}$$
$$\Rightarrow T - mg\sin\beta = m\frac{v^2}{3a}.$$

When applying the Principle of Conservation of Energy equation to compare the two points in the diagram we can consider the first point of the particle as being of height 0. As it is $3a\cos\theta = 2a$ below the ceiling, and the second point is of height $3a\sin\beta$ below the ceiling, it is of height $2a - 3a\sin\beta$ above the first point. Therefore we get

$$\begin{aligned}
\text{P.E.}_1 + \text{K.E.}_1 &= \text{P.E.}_2 + \text{K.E.}_2 \\
\Rightarrow 0 + \frac{m\sqrt{ag}^2}{2} &= mg(2a - 3a\sin\beta) + \frac{mv^2}{2} \\
\Rightarrow ag &= 4ag - 6ag\sin\beta + v^2 \\
\Rightarrow 6ag\sin\beta - 3ag &= v^2 \\
\Rightarrow 3ag(2\sin\beta - 1) &= v^2.
\end{aligned}$$

(ii) Continuing from part (i), substituting that value of v^2 into the centripetal force equation

$$\begin{aligned}
T - mg\sin\beta &= m\frac{v^2}{3a} \\
&= m\frac{3ag(2\sin\beta - 1)}{3a} \\
&= mg(2\sin\beta - 1) \\
&= 2mg\sin\beta - mg \\
\Rightarrow T &= 3mg\sin\beta - mg.
\end{aligned}$$

If we include all values of β, including when the particle is swinging back down, then we could have $\beta = 90^\circ$, in which case

$$\begin{aligned}
T &= 3mg(1) - mg \\
&= 2mg.
\end{aligned}$$

The minimum value of T happens when it is highest in its swing, i.e. when $v = 0$. If $v = 0$ then

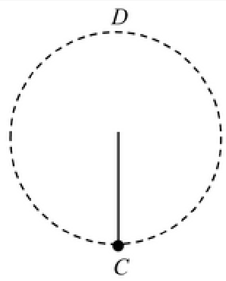
$$\begin{aligned}
3ag(2\sin\beta - 1) &= 0 \\
\Rightarrow \sin\beta &= \frac{1}{2} \\
\Rightarrow T &= 3mg\left(\frac{1}{2}\right) - mg \\
&= \frac{1}{2}mg.
\end{aligned}$$

Question — 2016 Q6 (a).

(a) A small particle hanging on the end of a light inextensible string 2 m long is projected horizontally from the point C.

(i) Calculate the least speed of projection needed to ensure that the particle reaches the point D which is vertically above C.

(ii) If the speed of projection is 7 m s^{-1} find the angle that the string makes with the vertical when it goes slack.



(i) Our points of interest are when the object is at the bottom and top of the circle. Assume the object has speed v at the top of the circle. Drawing the forces acting on the object at the top of the circle is quite straightforward.

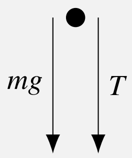


Figure 6

Therefore the centripetal force equation yields

$$F = m\frac{mv^2}{r}$$
$$\Rightarrow T + mg = \frac{mv^2}{2}.$$

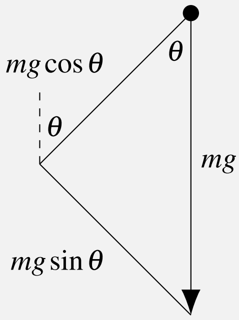
If the object is projected with the lowest possible speed so that it just about reaches the top of the circle then $T = 0$ at this point. Therefore

$$T + mg = \frac{mv^2}{2}$$
$$\Rightarrow mg = \frac{mv^2}{2}$$
$$\Rightarrow 2g = v^2.$$

Using the Principle of Conservation of Energy to compare the objects energy at the bottom and top of the circle, measuring potential energy from the bottom of the circle and giving an initial speed of u,

$$\begin{aligned}
\text{P.E.}_1 + \text{K.E.}_1 &= \text{P.E.}_2 + \text{K.E.}_2 \\
\Rightarrow 0 + \frac{mu^2}{2} &= mg(4) + \frac{mv^2}{2} \\
\Rightarrow u^2 &= 8g + v^2 \\
&= 8g + 2g \\
&= 10g \\
\Rightarrow u &= \sqrt{10g}.
\end{aligned}$$

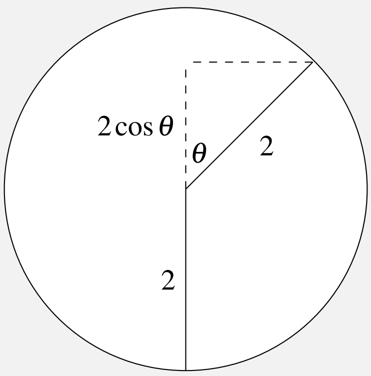
(ii) If instead $u = 7$, let θ be the angle that the string makes with the **upwards** vertical when the string goes slack. If the string goes slack when it makes an angle of θ with the upward vertical, then $T = 0$ at this point and only force acting on the particle is gravity, which looks as follows when resolved into and tangent to the circle.



Assuming a speed of v at this time, the centripetal force equation gives us

$$\begin{aligned}
F &= m\frac{v^2}{r} \\
\Rightarrow mg\cos\theta &= m\frac{v^2}{2} \\
\Rightarrow 2g\cos\theta &= v^2.
\end{aligned}$$

Regarding the Principle of Conservation of Energy, if we measure height from the bottom of the circle then when the string goes slack it is at a height of $2 + 2\cos\theta$.



Therefore comparing the energy this point with that of the point at the bottom of the circle,

$$\begin{aligned}
\text{P.E.}_1 + \text{K.E.}_1 &= \text{P.E.}_2 + \text{K.E.}_2 \\
\Rightarrow 0 + \frac{m(7)^2}{2} &= mg\,(2 + 2\cos\theta) + \frac{m(2g\cos\theta)}{2} \\
\Rightarrow \frac{49m}{2} &= 2mg + 2mg\cos\theta + mg\cos\theta \\
\Rightarrow \frac{49}{2} - 2g &= 3g\cos\theta \\
\Rightarrow \frac{1}{6} &= \cos\theta \\
\Rightarrow 80^\circ &= \theta.
\end{aligned}$$

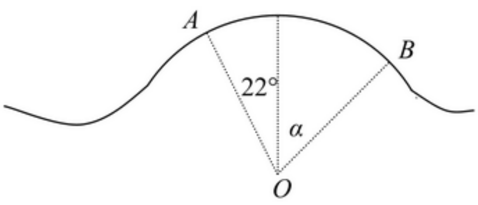
Question — 2015 Q6 (b).

(b) A skier of mass m kg is skiing on a hillside when he reaches a small hump in the form of an arc AB of a circle centre O and radius 7 m, as shown in the diagram.

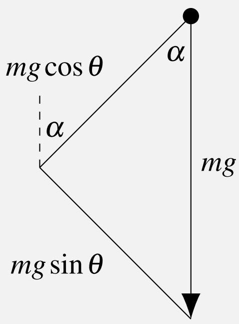
O, A and B lie in a vertical plane and OA and OB make angles of 22° and α with the vertical respectively.

The skier's speed at A is 8 m s^{-1}.

The skier looses contact with the ground at point B. Find the value of α.



Our two points of interest are A and B. Considering the centripetal force on the skier at B, as he loses contact with the sphere at this point the normal reaction force is equal to 0 and the only force acting on the skier is gravity, which looks as follows when resolved into and tangent to the circle.



Assuming a speed of v at this time, the centripetal force equation gives us

$$\begin{aligned} F &= m\frac{v^2}{r} \\ \Rightarrow mg\cos\alpha &= m\frac{v^2}{7} \\ \Rightarrow 7g\cos\alpha &= v^2. \end{aligned}$$

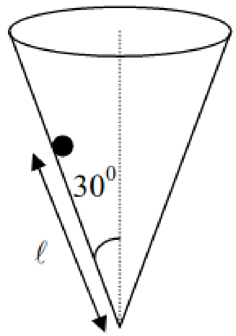
Using the Principle of Conservation of Energy to compare energies at A and B, measuring height from O,

$$\begin{aligned} \text{P.E.}_1 + \text{K.E.}_1 &= \text{P.E.}_2 + \text{K.E.}_2 \\ \Rightarrow mg(7\cos 22^\circ) + \frac{m(8)^2}{2} &= mg(7\cos\alpha) + \frac{mv^2}{2} \\ \Rightarrow 7g\cos 22^\circ + 32 &= 7g\cos\alpha + \frac{7g\cos\alpha}{2} \\ \Rightarrow 7g\cos 22^\circ + 32 &= 10.5g\cos\alpha \\ \Rightarrow \frac{7g\cos 22^\circ + 32}{10.5g} &= \cos\alpha \\ \Rightarrow 22^\circ &= \alpha. \end{aligned}$$

Question — 2006 Q6 (b).

A hollow cone with its vertex downwards and its axis vertical, revolves about its axis with a constant angular velocity of 4π rad/s.
A particle of mass m is placed on the inside rough surface of the cone. The particle remains at rest relative to the cone.

The coefficient of friction between the particle and the cone is $\frac{1}{4}$.
The semi-vertical angle of the cone is $30°$ and the particle is a distance ℓ m from the vertex of the cone.



Find the maximum value of ℓ, correct to two places of decimals.

If the particle remains at rest relative to the cone then it is actually moving at an angular speed of 4π rad/sec. If the radius of the circle is r then

$$\sin 30° = \frac{r}{l}$$
$$\Rightarrow \frac{l}{2} = r.$$

We will use r and ω later when we consider centripetal force.

We will assume that the the object is as high as possible without slipping. Therefore it is on the verge of slipping up but is not, and so friction is acting down the cone with the limiting friction is in play. This gives the following forces acting on the particle, resolved **horizontally and vertically**.

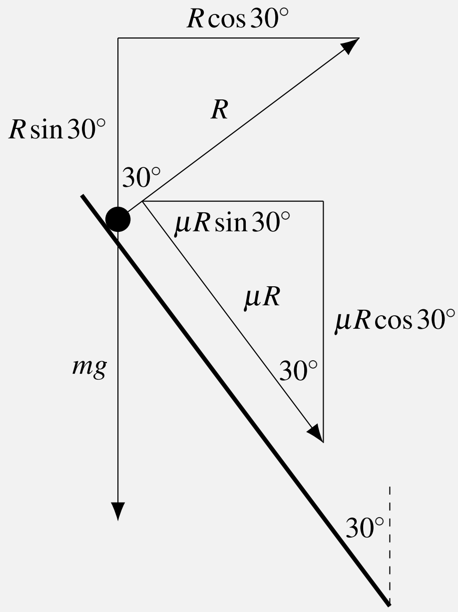


Figure 7

Letting our upwards and downwards forces be equal,

$$\begin{aligned} R\sin 30^\circ &= \mu R\cos 30^\circ + mg \\ \Rightarrow \frac{R}{2} &= \frac{1}{4}R\frac{\sqrt{3}}{2} + mg \\ \Rightarrow 4R - \sqrt{3}R &= 8mg \\ \Rightarrow R &= \frac{8mg}{4-\sqrt{3}}. \end{aligned}$$

Letting our inwards forces equal our centripetal force,

$$\begin{aligned} F &= m\omega^2 r \\ \Rightarrow R\cos 30^\circ + \mu R\sin 30^\circ &= m(4\pi)^2\frac{l}{2} \\ \Rightarrow R\left(\frac{\sqrt{3}}{2} + \frac{1}{8}\right) &= 8\pi^2 ml \\ \Rightarrow \frac{8mg}{4-\sqrt{3}}\frac{4\sqrt{3}+1}{8} &= 8\pi^2 ml \\ \Rightarrow \frac{(4\sqrt{3}+1)g}{(4-\sqrt{3})8\pi^2} &= l \\ \Rightarrow 0.43\text{ m} &\approx l. \end{aligned}$$

Question — 2000 Q6 (a).

(a) A particle is placed on a horizontal rotating turntable, 10 cm from the centre of rotation. There is a coefficient of friction of 0.4 between the particle and the turntable. If the speed of the turntable is gradually increased, at what angular speed will the particle begin to slide?

The two circular motion variables in play are angular velocity (as it's the answer) and $r = 0.1$ (remember that we have to measure in metres when forces are involved). Assume that the turntable is spun fast enough that the particle is about to slide but has not yet done so. Then the limiting friction is in play, acting towards the centre of the circle. If we imagine the view from behind the particle as it turns right, the following forces are acting on the particle at this time.

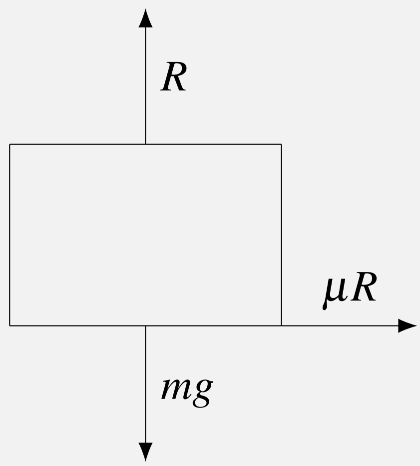


Figure 8

Letting upwards forces equal downwards forces, $R = mg$. Then our centripetal force equation becomes

$$\begin{aligned} F &= m\omega^2 r \\ \Rightarrow \mu R &= m\omega^2(0.1) \\ \Rightarrow 0.4mg &= 0.1m\omega^2 \\ \Rightarrow 4g &= \omega^2 \\ \Rightarrow \sqrt{4g} &= \omega. \end{aligned}$$

The particle will start to slide when ω increases beyond $\sqrt{4g}$ rad/sec.