Applied Mathematics

6th Year

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Introduction

This book is designed to be read as a single document, in the order in which the chapters appear. In each chapter questions, examples, definitions etc. are on a common numbering system, meaning that Definition 4.7 is the 7th text box in Chapter 4. Similarly figures are on a separate common numbering system, so that Figure 7.12 is the 12th picture in Chapter 7. This book also appears in pdf format on Moodle, in which all references are hyperlinked for easier navigation.

At the end of each chapter is a summary, which explains how the content of the chapter fits into the Leaving Cert syllabus and maps onto past (and possibly future) exam questions, as well as commenting on the differences between the old (pre-2023) and new syllabus. There is also a set of homework problems (separated by section) with solutions provided at the end of the chapter. This is to be completed as we work through the chapter. There is also a revision section at the end of each chapter for students re-reading the chapter or preparing for a test.

This book is comprehensive, assuming no knowledge of Applied Maths beyond this book and so you don't need any other book or notes to study for the course. It is also designed so that it can be read as revision weeks or months after we first cover them. As such you shouldn't need to spend much time taking notes and can instead concentrate on the class. However you may want to take notes occasionally if there is something mentioned in class that isn't covered clearly in the book.

Tests will take place shortly after we complete a chapter, and will be of a similar difficulty to homework and revision problems.

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Mechanics

1. Linear Motion

1.1 Some Initial Terminology

In this chapter we will consider motion in one dimension (i.e., a straight line) with constant acceleration. That is, if the speed of an object increases by 2 units in 1 second, it will increase by another 2 units in the next second, and indeed by 1 unit in the next half second.

In real life we often use terms like speed and distance. However in Science we use the more precise terminology of velocity and displacement. This is because in Science we differentiate between **vectors** and **scalars**.

Definition 1.1

- A **scalar** is a physical quantity that has magnitude only.
- A **vector** is a quantity that has magnitude and direction.
- The **distance** travelled by an object, or between two objects, is a scalar.
- The **displacement** of an object is its distance from a fixed point in a given direction, and so is a vector.
- The **speed** of an object is a scalar.
- The **velocity** of an object is its speed in a given direction, and so is a vector.
- The **acceleration** of an object is the rate of change of its velocity, and so is a vector.
- If an object has negative acceleration, it has positive **deceleration** (occasionally called **retardation**) of the same magnitude.

For example, saying that Belfast is 150 km from Dublin is giving the scalar **distance** between them. Saying that Belfast is 150 km **North** of Dublin is giving the **displacement** of Belfast from the fixed point that is Dublin. Similarly, saying that you fired a cannonball at 100 km/hr is giving the **speed** of the cannonball, but saying that you fired the cannonball at a speed of 100 km/hr and **at an angle of** $30°$ **to the horizontal** is giving a direction and therefore is giving the velocity of the cannonball.

As we are dealing only with one-dimensional motion in this chapter, the way in which we give direction will be quite straight-forward: one direction will be considered the positive direction, just like on a number line or on a coordinate geometry axis. Velocities and displacements will be positive if they are in that direction, and negative if they are in the opposite direction. One point, usually the point where we begin the journey, will be considered the fixed point from which displacement is measured and can be thought of as the origin of this number line.

In the diagram below there are four people, Alice, Bob, Charlie and Daphne, each represented by a dot, with arrows showing their direction of motion. The small black dot is the point from which we measure displacement. We will consider the **rightwards** direction as positive.

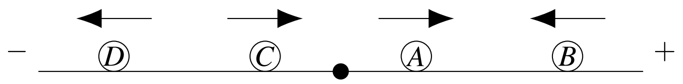


Figure 1.1

We don't have any units or quantities, but Alice is to the right of the fixed point, so her displacement is positive. She is also moving to the right, so her velocity is positive. Bob also has a positive displacement, but his velocity is negative. Charlie has a negative displacement but a positive velocity and Daphne has a negative velocity and displacement.

Moving on to acceleration, consider Alice's position and velocity at two different times. Say she is moving to the right, but her positive velocity is increasing (as indicated by the bigger arrow).

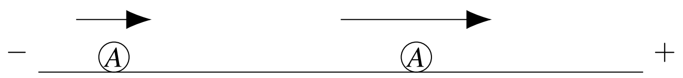


Figure 1.2

As her velocity is **increasing**, her acceleration is **positive**. On the other hand, Daphne's position and velocity at two different times as she moves to the left are also shown below.

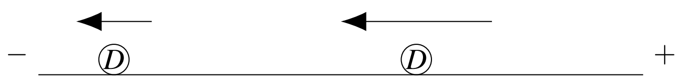


Figure 1.3

Her velocity is changing from a large negative number to a smaller negative number, and so is also **increasing**.

> **Note 1.2** In both cases, we can imagine the acceleration the girls are experiencing is caused by something, for example wind. If so the wind is travelling to the right, *pushing* the girls to the right. The idea that positive acceleration is caused by something, in particular by a push in the positive direction is a true one that we will study in more depth in Chapter 2.

Now consider the position and velocity of Bob at two different times below as he moves to the left.

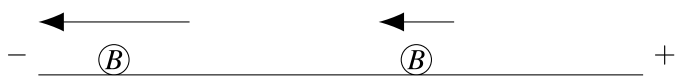


Figure 1.4

Bob has a negative velocity initially, which grows larger (becoming a larger negative number) and so is **decreasing**. This means Bob has a negative acceleration, or that Bob is **decelerating**.

Finally, consider the position and velocity of Charlie at two different times below as he moves to the left.

Figure 1.5

Charlie has a positive velocity initially, which grows smaller and so is **decreasing**. This means Charlie also has a negative acceleration, or that Charlie is **decelerating**.

Note 1.3 Applying a similar logic to that in Note 1.2, we can imagine that the boys' deceleration is caused by something, again perhaps wind. In this case the wind is pushing to the left, i.e. in the negative direction, which causes deceleration.

Note 1.4 This exploration of Alice, Bob, Charlie and Daphne's movement can be summarised as follows. Say the positive direction is to the right. If an object is to the right (of the fixed point), its displacement is positive. If it is moving to the right, its velocity is positive, and if it has acceleration pushing it to the right its acceleration is positive.

Finally, when we do consider units and measurements, in this chapter there is a simple relationship between speed and velocity, and between displacement and distance.

Rule 1.5 In the case of Linear Motion, if an object has velocity v, it has speed $|v|$. If an object has displacement s from a fixed point, its distance from that fixed point is $|s|$.

For example, if in Figure 1.1 we were measuring in metres (m) and seconds (s) and Daphne was 5 m to the left of the fixed point with a speed of 2 m/s, her displacement would be -5 m and her velocity would be -2 m/s. If Alice was 5 m to the right of the fixed point with a speed of 2 m/s, her displacement would simply be 5 m and her velocity would simply be 2 m/s.

Note 1.6 In reality, most Linear Motion problems will have the fixed point from which we measure displacement be the starting point of the journey, and many Linear Motion problems won't have a change in direction. This means that we can take whatever direction the object is moving in for the entire journey be the positive direction. It is only where objects change direction where we need to consider which direction is the positive direction. These are most common in freefall problems, which we will study in Section 1.3.

1.2 Five Quantities, Five Equations

Now that we've covered basic concepts and terminology we'll move on a more numerical study. In Linear Motion problems, for each object in question we are interested in 5 quantities:

- u: the initial velocity of the object,
- v: the final velocity of the object,
- a: the acceleration of the object,
- s: the displacement of the object from its initial position,
- t: the time of the journey.

In most questions displacement will be measured in metres and time in seconds, and therefore velocity in m/s (occasionally written ms^{-1}) and acceleration in m/s^2 (occasionally written ms^{-2}).

Occasionally we will instead measure in km and hours. It is important when giving answers to questions to include units.

Generally we will be given incomplete information about the situation, and will need to infer the rest ourselves. These five quantities are related by the following five equations.

Rule 1.7 — The Equations of Motion. If, over t seconds, an object has uniform acceleration a, an initial velocity u, a final velocity v and a final displacement of s from its starting point,

$$v = u + at$$
$$s = \left(\frac{u+v}{2}\right)t$$
$$v^2 = u^2 + 2as$$
$$s = ut + \frac{1}{2}at^2$$
$$s = vt - \frac{1}{2}at^2$$

The first four of these formulae appear in the following form on page 50 of *Formulae & Tables*. However the fifth **does not**.

Meicnic		**Mechanics**
fórsa agus luasghéarú	$F = ma$	force and acceleration
Gluaisne líneach faoi luasghéarú tairiseach	$v = u + at$	**Linear motion with constant acceleration**
	$s = ut + \frac{1}{2}at^2$	
	$v^2 = u^2 + 2as$	
	$s = \left(\frac{u+v}{2}\right)t$	

Figure 1.6

Note 1.8 It is perfectly fine to use the fifth equation of motion when answering questions in this chapter, and in exams. In fact it makes answering some past exam questions significantly easier as we will see in Section 1.10. However it is not given in the *Formulae & Tables* and so if you want to use it you need to have it memorised yourself.

Note 1.9 These equations can be deduced individually. The first is a rearraging of the definition of acceleration, written without fractions:

$$a = \frac{v-u}{t}.$$

The second equation is more of a common sense deduction. If the velocity changes at a constant rate from u to v then the average velocity is $\frac{u+v}{2}$, and displacement would equal average velocity multiplied by time.

The third equation comes from rearranging the first equation as

$$\frac{v-u}{a} = t$$

and substituting it into the second equation.

$$\begin{aligned} s &= \left(\frac{u+v}{2}\right)t \\ &= \left(\frac{u+v}{2}\right)\left(\frac{v-u}{a}\right) \\ &= \frac{uv-u^2+v^2-uv}{2a} \\ \Rightarrow 2as &= -u^2+v^2 \\ \Rightarrow u^2+2as &= v^2. \end{aligned}$$

The fourth equation similarly comes from replacing v with $u+at$ in the second equation.

$$\begin{aligned} s &= \left(\frac{u+v}{2}\right)t \\ &= \left(\frac{u+u+at}{2}\right)t \\ &= \left(u+\frac{1}{2}at\right)t \\ &= ut+\frac{1}{2}at^2. \end{aligned}$$

The fourth equation comes from rearranging the first equation as

$$u = v - at$$

and substituting it into the second equation.

$$\begin{aligned} s &= \left(\frac{u+v}{2}\right)t \\ &= \left(\frac{v-at+v}{2}\right)t \\ &= \left(v-\frac{1}{2}at\right)t \\ &= vt-\frac{1}{2}at^2. \end{aligned}$$

Note 1.10 Notice that each of the five equations of motion contain four of the five variables of motion. Given any **three** of the variables u, v, a, s and t we can use these equations to find the other two by choosing an equation with three variables we know and one we want to find.

Example 1.11 A car passes a set of traffic lights at 10 m/s, and accelerates at 2 m/s^2 until it passes a speed camera 75 m away. What is its speed as it passes the speed camera? How long does the journey take?

Writing down the variables we know, we get

$$\begin{aligned} u &= 10 \\ v &= \\ a &= 2 \\ s &= 75 \\ t &= \end{aligned}$$

This is called a UVAST array, and we will use them throughout this chapter. In solving problems, you can fill in the UVAST array as you discover values, although you won't always see that in these notes. To find v or t, we need to choose one of the 4 equations we constructed that contains only one of the variables we don't know. For example, $v^2 = u^2 + 2as$,

$$\begin{aligned} v^2 &= u^2 + 2as \\ \Rightarrow v^2 &= 10^2 + 2(2)(75) \\ &= 400 \\ \Rightarrow v &= 20 \text{ m/s.} \end{aligned}$$

Once we know four variables, we can find the fifth using any equation that contains it.

$$\begin{aligned} s &= \left(\frac{u+v}{2}\right)t \\ \Rightarrow 75 &= 15t \\ \Rightarrow t &= 5 \text{ seconds.} \end{aligned}$$

Note 1.12 It is important to include units in all answers where relevant.

Example 1.13 A cyclist, accelerating uniformly from rest, covers 100 m in 20 seconds. What was his final speed and acceleration?

In this case our UVAST array is

$$\begin{aligned} u &= 0 \\ v &= \\ a &= \\ s &= 100 \\ t &= 20 \end{aligned}$$

Note that starting "from rest" means starting with a speed of 0 m/s. Using $s = \left(\frac{u+v}{2}\right)t$ we have

$$\begin{aligned} s &= \left(\frac{u+v}{2}\right)t \\ \Rightarrow 100 &= \left(\frac{0+v}{2}\right)20 \\ &= 10v \\ \Rightarrow 10 \text{ m/s} &= v. \end{aligned}$$

Then

$$\begin{aligned} v &= u+at \\ \Rightarrow 10 &= 0+20a \\ \Rightarrow \frac{1}{2} \text{ m/s}^2 &= a. \end{aligned}$$

Example 1.14 A motorist is travelling at 20 m/s and, noticing a speed camera 150 m away, decelerates uniformly so that her speed is 10 m/s just as she passes the speed camera. What was her deceleration?

In this case our UVAST array is

$$\begin{aligned} u &= 20 \\ v &= 10 \\ a &= \\ s &= 150 \\ t &= \end{aligned}$$

We're not asked for t, only a, so

$$\begin{aligned} v^2 &= u^2+2as \\ \Rightarrow 10^2 &= 20^2+2a(150) \\ \Rightarrow 100 &= 400+300a \\ \Rightarrow -300 &= 300a \\ \Rightarrow -1 \text{ m/s}^2 &= a. \end{aligned}$$

If $a = -1$ then the **deceleration** of the motorist is $+1$ m/s^2.

Note 1.15 In practice there won't be much ambiguity in how you should give your answer when asked for deceleration, but it is important to understand that a deceleration of $+1$ and an acceleration of -1 are the same.

Note 1.16 Notice that these questions used the terms speed and distance rather than velocity and displacement. This is common in questions, but for now the distinction is unimportant as the objects don't change direction.

Question 1.17 A bus is travelling at a velocity of 20 m/s, and anticipating the bus stop 100 m ahead begins to decelerate uniformly until it comes to rest just at the bus stop. What is its uniform deceleration? How long does it take to get to the bus stop?

Question 1.18 A motorbike accelerates from 20 m/s to 30 m/s at a rate of 5 m/s^2. How much distance does it cover in this time?

Question 1.19 A cyclist, noticing a road blockage ahead decelerates at a rate of 2 m/s^2 until he comes to rest 25 m from his starting point. What was his original speed?

1.3 Freefall

A special case of uniform acceleration is acceleration due to gravity. When in the air, a solid object will fall with a constant acceleration of $g = 9.8$ m/s^2 (we ignore wind resistance). This value is given on the cover page of the exam. In questions involving gravity, we will often (but not always) let the upwards direction be the positive direction. In that case velocity will be positive until the object reaches its highest point, after which it will be negative.

Note 1.20 In freefall problems, the following is true.

- If an object is "dropped", its initial velocity is 0.
- When an object is at its greatest height its velocity is 0.
- If an object is thrown from ground level, when it hits the ground again its displacement is 0.
- If an object is thrown from a height h above ground level, when it hits the ground again its displacement is $-h$ (if we count the upwards direction as positive).

Example 1.21 An object is projected vertically upwards into the air with a velocity of 40 m/s. How long will it take to reach its highest point? How high in the air will it reach?

When an object reaches its highest point its velocity is zero. So writing out our variables we have

$$\begin{aligned} u &= 40 \\ v &= 0 \\ a &= -9.8 \\ s &= \\ t &= \end{aligned}$$

Notice that u is positive as the object is initially moving upwards, but a is negative as gravity is

pulling the object down.

$$
\begin{aligned}
v &= u + at \\
\Rightarrow 0 &= 40 - 9.8t \\
\Rightarrow t &= 4.08 \text{ seconds} \\
\Rightarrow s &= \left(\frac{u+v}{2}\right)t \\
&= 81.6 \text{ m.}
\end{aligned}
$$

As you can see here, the introduction of the decimal for acceleration can cause awkward decimal answers. In some questions, you will be asked to give an answer to one or two decimal places. However in many cases you will be asked to give your answer in terms of g. Therefore you will leave g as a variable, like you often do with π in area and volume problems. Even if not explicitly asked to do this it can be easier to do so, and is good practice for harder problems later. Repeating the problem in this way, we have

$$
\begin{aligned}
u &= 40 \\
v &= 0 \\
a &= -g \\
s &= \\
t &=
\end{aligned}
$$

Then we get

$$
\begin{aligned}
v &= u + at \\
\Rightarrow 0 &= 40 - gt \\
\Rightarrow t &= \frac{40}{g} \text{ seconds} \\
\Rightarrow s &= \left(\frac{u+v}{2}\right)t \\
&= \left(\frac{40+0}{2}\right)\frac{40}{g} \\
&= \frac{800}{g} \text{ m.}
\end{aligned}
$$

Note 1.22 In conclusion, we will leave g as a variable, like we often do with π in Area and Volume problems, unless it is far too complicated to do so or we are asked not to.

Example 1.23 An object is dropped from the top of a 10 m cliff. What is its speed when it hits the ground?

When the object hits the ground it is 10 m below where it started, so we could say that $s = -10$ at the end of this journey. Also, if an object is "dropped" then its initial velocity is 0. Therefore we could write our UVAST array as

$$\begin{aligned} u &= 0 \\ v &= \\ a &= -g \\ s &= -10 \\ t &= \end{aligned}$$

Then

$$\begin{aligned} v^2 &= u^2 + 2as \\ &= 0^2 + 2(-g)(-10) \\ &= 20g \\ \Rightarrow v &= -\sqrt{20g} \\ &= -14 \text{ m/s}. \end{aligned}$$

We know velocity is negative because the object is travelling downwards. However, we were asked for the **speed** of the object, and so our answer is that the speed of the object is 14 m/s.

As an alternative way of solving this problem, as all movement is downwards, we could have set the downwards direction as positive so that our UVAST array was

$$\begin{aligned} u &= 0 \\ v &= \\ a &= g \\ s &= 10 \\ t &= \end{aligned}$$

and

$$\begin{aligned} v^2 &= u^2 + 2as \\ &= 0^2 + 2(g)(10) \\ &= 20g \\ \Rightarrow v &= \sqrt{20g} \\ &= 14 \text{ m/s}. \end{aligned}$$

Note 1.24 There are many points to note regarding this example.

1. This example shows the importance of choosing which direction is the positive direction. There is never any wrong decision when it comes to choosing the positive direction, but if an object does not change direction then it is much simpler if the direction it travels in is chosen to be the positive direction.

2. You will see throughout this course that multiples of g can give nice answers when we take square roots. In particular $\sqrt{5g} = 7$ and $\sqrt{\frac{g}{5}} = 1.4$. When tackling longer problems it is advised not to replace exact expressions like $\sqrt{2}$ with decimals; however it is worth

checking that square roots involving g do not simplify to a whole number or terminating decimal. It is much nicer to have a quantity equal to 14 than $\sqrt{20g}$ when using it in further calculations.

3. Note that our question asked for speed, not velocity. Although sometimes there is no difference, for example in our second attempt both speed and velocity are equal to 14, it is unwise to give a negative answer if asked for speed. Exam questions on Linear Motion generally ask for speed and distance rather than velocity and displacment as speed and distance are unambiguous but the sign of velocty and displacement depend on which direction the student decided was the positive direction. On the other hand, when taking a square root when using $v^2 = u^2 + 2as$ it is necessary to know whether your velocity should be positive or negative if it is not a final answer but instead an intermediary calculation.

4. On a similar note, questions in which the object changes direction may ask for the total distance travelled which is not as simple as the absolute value of the displacement (see Question 1.27).

Question 1.25 An object is thrown directly upwards from the ground, reaching a maximum height of 44.1 m. At what initial velocity was the object thrown at?

Question 1.26 An object is thrown upwards from the ground with an initial velocity of 30 m/s. What is its velocity after four seconds? What is its speed? In what direction is it travelling at this time?

Question 1.27 An object is thrown directly upwards from a balcony that is 20 m above the ground, at a speed of 25 m/s. What is its greatest height above the ground? How long does the object take to hit the ground, and at what speed does it do so? What is the total distance travelled by the object?

Note 1.28 Notice in Question 1.27 that we needed to consider two journeys. To answer the first question, the journey is from the balcony to the greatest height, and to answer the second, the journey is from the balcony to the ground. Often in freefall problems, by declaring a displacement or final velocity you are deciding the part of the journey that you are studying.

1.4 Multi-Part Journeys

We will now look at situations where acceleration is not constant over the whole journey, but instead changes instantly at specified points in time.

Example 1.29 A car starts from rest and accelerates at 1 m/s^2 for 10 seconds. It then decelerates at 2 m/s^2 until it comes to rest. How long does the car spend decelerating?

The trick in this question, and indeed in all questions of this type, is to note that the final velocity in the first part of the journey is equal to the initial velocity in the second part of the journey. For these questions we will write a UVAST column for each part of the journey in the following way:

First Part	Second Part	Extra Equations
$u_1 = 0$	$u_2 =$	$v_1 = u_2$
$v_1 =$	$v_2 = 0$	
$a_1 = 1$	$a_2 = -2$	
$s_1 =$	$s_2 =$	
$t_1 = 10$	$t_2 =$	

To be clear, the displacements s_1 and s_2 are the total displacement over each part of the journey, meaning that the reference point is the position of the object at the start of this part of the journey. Similarly, the "initial" velocity u_2 is the velocity at the start of the second part of the journey.

We only know two variables for the second part of the journey, so we cannot find t_2 immediately. But since we know $u_2 = v_1$, we can find u_2 by finding v_1.

$$\begin{aligned} v_1 &= u_1 + a_1 t_1 \\ &= 10, \end{aligned}$$

and so $u_2 = 10$. Now knowing u_2, v_2 and a_2 we can find t_2.

$$\begin{aligned} v_2 &= u_2 + a_2 t_2 \\ \Rightarrow 0 &= 10 - 2t \\ \Rightarrow t &= 5 \text{ seconds.} \end{aligned}$$

In the Applied Mathematics exam multi-part journeys will usually consist of 2-3 parts; in the 3-part case we will solve the problem by letting $u_2 = v_1$, and $u_3 = v_2$.

Example 1.30 A car passes a point P at 12 m/s, accelerating at a uniform rate of 6 m/s^2 until it reaches a speed of 30 m/s. It continues at this speed for 12 seconds and then slows to rest at a point Q in 5 seconds. What is its average speed for the entire journey from P to Q?

In this case our UVAST array look as follows:

First Part	Second Part	Third Part
$u_1 = 12$	$u_2 = 30$	$u_3 = 30$
$v_1 = 30$	$v_2 = 30$	$v_3 = 0$
$a_1 = 6$	$a_2 = 0$	$a_3 =$
$s_1 =$	$s_2 =$	$s_3 =$
$t_1 =$	$t_2 = 12$	$t_3 = 5$

In this case we already have that $v_1, u_2, v_2, u_3 = 30$ so we don't any extra equations. We get the average speed over an entire trip from dividing the total distance travelled by the total time, i.e. in this case

$$\text{Average Speed} = \frac{s_1 + s_2 + s_3}{t_1 + t_2 + t_3}.$$

We can deal with the UVAST columns in any order to fully fill them in as we have three quantities in each column. Starting with the first,

$$\begin{aligned} v_1 &= u_1 + a_1 t_1 \\ \Rightarrow 30 &= 12 + 6t_1 \\ \Rightarrow 3 \text{ seconds} &= t_1. \\ s &= \left(\frac{u+v}{2}\right)t \\ \Rightarrow s_1 &= \left(\frac{12+30}{2}\right)3 \\ &= 63 \text{ m.} \end{aligned}$$

Then

$$\begin{aligned} s_2 &= \left(\frac{u_2+v_2}{2}\right)t_2 \\ \Rightarrow s_2 &= \left(\frac{30+30}{2}\right)12 \\ &= 360 \text{ m.} \end{aligned}$$

In the final column,

$$\begin{aligned} s_3 &= \left(\frac{u_3+v_3}{2}\right)t_3 \\ \Rightarrow s_3 &= \left(\frac{30+0}{2}\right)5 \\ &= 75 \text{ m.} \end{aligned}$$

Therefore

$$\begin{aligned} \text{Average Speed} &= \frac{s_1+s_2+s_3}{t_1+t_2+t_3} \\ &= \frac{63+360+75}{3+12+5} \\ &= 24.9 \text{ m/s.} \end{aligned}$$

Note 1.31 Notice in this example I wrote $v_1 = u_1 + a_1 t_1$ rather than $v = u + at$ when using the equation $v = u + at$ on the first column, and similar in other cases. This makes it more clear to other readers, and will be more important when we deal with more algebraic problems in other chapters, so it's recommended.

Question 1.32 A car starts from rest to a speed of 40 m/s with an acceleration of 5 m/s^2, continues at a uniform speed for 15 seconds, and then slows to rest with a deceleration of 4 m/s^2.

(a) What is the total time of the journey?

(b) What is the total distance travelled?

(c) What is the average speed over the whole journey?

Question 1.33 A car passes a point P at 14 m/s, accelerating at a rate of 3 m/s^2 for 12 seconds. It travels at a constant speed for some time before decelerating uniformly to rest over 10 seconds, stopping at the point Q. The total distance from P to Q is 884 m. How long does the car spend travelling at a constant speed?

1.5 More Complex Multi-Part Journeys

The multi-part journeys in Section 1.4 serve as an introduction to the basic mechanics of multi-part journeys. However many problems are not as straight-forward as those.

Example 1.34 A cyclist, starting from rest, accelerates at a uniform rate of 3 m/s^2. She then immediately decelerates to rest at a uniform rate of 2 m/s^2. The total time of the journey was 25 seconds. Find the maximum speed of the cyclist.

Based on previous approaches, our UVAST array looks as follows.

First Part	Second Part	Extra Equations
$u_1 = 0$	$u_2 =$	$v_1 = u_2$
$v_1 =$	$v_2 = 0$	$t_1 + t_2 = 25$
$a_1 = 3$	$a_2 = -2$	
$s_1 =$	$s_2 =$	
$t_1 =$	$t_2 =$	

We only know two values in each column and so can't find the values of any more at first. What we will do instead is first **convert the extra equations to algebra**. If $v_1 = u_2$, we can let $v_1, u_2 = v$ in the UVAST array, and discard that extra equation.

First Part	Second Part	Extra Equations
$u_1 = 0$	$u_2 = v$	~~$v_1 = u_2$~~
$v_1 = v$	$v_2 = 0$	$t_1 + t_2 = 25$
$a_1 = 3$	$a_2 = -2$	
$s_1 =$	$s_2 =$	
$t_1 =$	$t_2 =$	

Similarly, we can let $t_1 = t$, so that

$$\begin{aligned} t_2 &= 25 - t_1 \\ &= 25 - t. \end{aligned}$$

Therefore our UVAST array looks as follows.

First Part	Second Part	Extra Equations
$u_1 = 0$	$u_2 = v$	~~$v_1 = u_2$~~
$v_1 = v$	$v_2 = 0$	~~$t_1 + t_2 = 25$~~
$a_1 = 3$	$a_2 = -2$	
$s_1 =$	$s_2 =$	
$t_1 = t$	$t_2 = 25 - t$	

We haven't found the values of any other quantities in the UVAST array, but we can set up two simultaneous equations in v and t.

$$\begin{aligned} v_1 &= u_1 + a_1 t_1 \\ \Rightarrow v &= 3t. \\ v_2 &= u_2 + a_2 t_2 \\ \Rightarrow 0 &= v - 2(25 - t) \\ \Rightarrow 0 &= v - 50 + 2t \\ \Rightarrow 50 - 2t &= v. \end{aligned}$$

Setting both v's equal to each other,

$$\begin{aligned} 3t &= 50 - 2t \\ \Rightarrow 5t &= 50 \\ \Rightarrow t &= 10 \\ \Rightarrow v &= 3(10) \\ &= 30 \text{ m/s}. \end{aligned}$$

Note 1.35 There was no particular reason I let $t_1 = t$ and not t_2, it can be done the other way around.

Note 1.36 Back in Section 1.4, in Note 1.31 it was mentioned how it was recommended to use subscripts when writing down the initial equation of motion to be used. This should be more clear now. In the previous example I wrote

$$\begin{aligned} v_2 &= u_2 + a_2 t_2 \\ \Rightarrow 0 &= v - 2(25 - t). \end{aligned}$$

If instead I wrote

$$\begin{aligned} v &= u + at \\ \Rightarrow 0 &= v - 2(25 - t), \end{aligned}$$

then among other things it would be confusing as to why t got replaced with $25 - t$. In fact some students would forget that it is $25 - t$ and not t itself that goes in that position in the equation. While students wouldn't get penalised for not using subscripts in an exam, they are likely to confuse themselves if they don't use subscripts when setting up an equation.

To formalise this setup, we have the following approach.

Rule 1.37 When trying to solve a problem with a UVAST array with multiple columns and extra equations, take the following approach.

1. For each extra equation with two variables, replace the extra equation with algebra by setting one subscript variable equal to a letter variable, and writing the other subscript variable in terms of it (we will deal with extra equations with more than two variables in Note 1.58).
2. Place the algebraic form of the subscript variables in the UVAST array.

3. By either setting up simultaneous equations or finding each variable one by one in terms of the other remaining variables, solve for the variables in the UVAST array. Do this by applying the equations of motion to each column, in the following way.
   - Don't apply the same equation of motion more than once to the same column.
   - If a column is filled entirely with variables, numbers and $a \neq 0$, you can apply two equations to the same column. It doesn't matter what the second one is as long as it is not the same as the first one.
   - The number of equations of motion you can apply to a column is at most 2. Subtract 1 for each empty subscripted variable in the column, and subtract another 1 if $a = 0$.
4. From there find the answer that the question is asking for, which may not be the value of any the variables used to replace the extra equations.

Note 1.38 The mention of how to apply the equations of motion to step 3 is useful so that in an exam situation you don't run around in circles applying the same equations and repeating the same work.

Example 1.39 An object accelerates from rest to a speed of 10 m/s, then immediately decelerates to rest at three times the previous rate of acceleration. The object travels 100 m in total. How long did the journey take?

Our UVAST array looks as follows.

First Part	Second Part	Extra Equations
$u_1 = 0$	$u_2 = 10$	$s_1 + s_2 = 100$
$v_1 = 10$	$v_2 = 0$	$a_2 = -3a_1$
$a_1 =$	$a_2 =$	
$s_1 =$	$s_2 =$	
$t_1 =$	$t_2 =$	

Letting $a_1 = a$ so that $a_2 = -3a$, and $s_1 = s$ so that $s_2 = 100 - s$, the UVAST array looks as follows.

First Part	Second Part	Extra Equations
$u_1 = 0$	$u_2 = 10$	~~$s_1 + s_2 = 100$~~
$v_1 = 10$	$v_2 = 0$	~~$a_2 = -3a_2$~~
$a_1 = a$	$a_2 = -3a$	
$s_1 = s$	$s_2 = 100 - s$	
$t_1 =$	$t_2 =$	

To set up simultaneous equations we have to use $v^2 = u^2 + 2as$.

$$\begin{aligned} v_1^2 &= u_1^2 + 2a_1 s_1 \\ \Rightarrow 10^2 &= 0^2 + 2as \\ \Rightarrow 100 &= 2as \\ \Rightarrow 50 &= as. \\ v_2^2 &= u_2^2 + 2a_2 s_2 \\ \Rightarrow 0^2 &= 10^2 - 6a(100 - s) \\ \Rightarrow 0 &= 100 - 600a + 6as \\ \Rightarrow 0 &= 50 - 300a + 3as. \end{aligned}$$

Our system of equations is

$$\begin{aligned} 50 &= as, \\ 0 &= 50 - 300a + 3as. \end{aligned}$$

It's easiest just to use the first equation to replace *as* in the second equation.

$$\begin{aligned} 0 &= 50 - 300a + 3as \\ \Rightarrow 0 &= 50 - 300a + 150 \\ \Rightarrow 300a &= 200 \\ \Rightarrow a &= \frac{2}{3}. \end{aligned}$$

Then

$$\begin{aligned} 50 &= as \\ \Rightarrow 50 &= \left(\frac{2}{3}\right) s \\ \Rightarrow 75 &= s. \end{aligned}$$

While this work is useful, we are not done as we were asked for the total time $t_1 + t_2$. Since we know a and s this is now easy. It may be helpful to fill in these values into the UVAST array if you're working on paper.

First Part	Second Part	Extra Equations
$u_1 = 0$	$u_2 = 10$	$\cancel{s_1 + s_2 = 100}$
$v_1 = 10$	$v_2 = 0$	$\cancel{a_2 = -3a_2}$
$a_1 = \cancel{a}\ \frac{2}{3}$	$a_2 = \cancel{-3a}\ -2$	
$s_1 = \cancel{s}\ 75$	$s_2 = \cancel{100 - s}\ 25$	
$t_1 =$	$t_2 =$	

We know four quantities in each column, making it easy to find that $t_1 = 15$, $t_2 = 5$ so that

$$\begin{aligned} \text{Answer} &= t_1 + t_2 \\ &= 20 \text{ seconds.} \end{aligned}$$

Note 1.40 Regarding Step 3 of Rule 1.37, notice how each column had one blank subscripted variable and $a \neq 0$, so we could only apply one equation to each column.

Example 1.41 A sprinter runs a race with constant acceleration. During the race he passes four posts, A, B, C and D, with $|AB| = |BC| = |CD| = 20$ m. If the sprinter takes 4 seconds to get from A to B and 3 seconds to get from B to C find how long it takes him to get from C to D.

We don't actually consider the journey from the beginning of the race to A, as we know little about it. The first part of our journey is $A \to B$, and so $u_1 \neq 0$.

First Part	Second Part	Third Part	Extra Equations
$u_1 =$	$u_2 =$	$u_3 =$	$u_2 = v_1$
$v_1 =$	$v_2 =$	$v_3 =$	$u_3 = v_2$
$a_1 =$	$a_2 =$	$a_3 =$	$a_1 = a_2 = a_3$
$s_1 = 20$	$s_2 = 20$	$s_3 = 20$	
$t_1 = 4$	$t_2 = 3$	$t_3 =$	

Converting our extra equations to algebra by letting $u_2, v_1 = v$, $u_3, v_2 = w$, and $a_1, a_2, a_3 = a$, we get

First Part	Second Part	Third Part	Extra Equations
$u_1 =$	$u_2 = v$	$u_3 = w$	~~$u_2 = v_1$~~
$v_1 = v$	$v_2 = w$	$v_3 =$	~~$u_3 = v_2$~~
$a_1 = a$	$a_2 = a$	$a_3 = a$	~~$a_1 = a_2 = a_3$~~
$s_1 = 20$	$s_2 = 20$	$s_3 = 20$	
$t_1 = 4$	$t_2 = 3$	$t_3 =$	

We can either set up three equations in three variables (**Method 1**), or remove the variables from the array one by one (**Method 2**).

Method 1: Starting with the first column,

$$\begin{aligned} s_1 &= v_1 t_1 - \frac{1}{2} a_1 t_1^2 \\ \Rightarrow 20 &= 4v - 8a. \end{aligned}$$

In the second column, it's better to avoid equations with squares if doing simultaneous equations, and also to have fewer variables if possible by using the known s_2 and t_2.

$$\begin{aligned} s_2 &= \left(\frac{u_2 + v_2}{2}\right) t_2 \\ \Rightarrow 20 &= \left(\frac{v + w}{2}\right) 3 \\ \Rightarrow \frac{40}{3} &= v + w. \end{aligned}$$

We can't do anything with the third column as we don't want to use the unknown v_3 or t_3. In the first column we don't want to use the unknown u_1, so returning to the second column and using a second equation,

$$\begin{aligned} v_2 &= u_2 + a_2 t_2 \\ \Rightarrow w &= v + 3a. \end{aligned}$$

Our system of simultaneous equations is then

$$\begin{aligned} 20 &= 4v - 8a \\ \frac{40}{3} &= v + w \\ w &= v + 3a \end{aligned}$$

Substituting w out of the second equation using the third equation we get

$$\begin{aligned} \frac{40}{3} &= v + v + 3a \\ \Rightarrow \frac{40}{3} &= 2v + 3a, \end{aligned}$$

giving us the reduced system

$$\begin{aligned} 20 &= 4v - 8a \\ \frac{40}{3} &= 2v + 3a \end{aligned}$$

Solving this system by writing the first equation with v as the subject and substituting it into the second,

$$\begin{aligned} 20 &= 4v - 8a \\ \Rightarrow 20 + 8a &= 4v \\ \Rightarrow 5 + 2a &= v \\ \Rightarrow \frac{40}{3} &= 2(5 + 2a) + 3a \quad \text{(substituting into second equation)} \\ \Rightarrow \frac{40}{3} &= 10 + 4a + 3a \\ \Rightarrow \frac{10}{3} &= 7a \\ \Rightarrow \frac{10}{21} &= a. \end{aligned}$$

Then

$$\begin{aligned} v &= 5 + 2a \\ \Rightarrow v &= 5 + 2\left(\frac{10}{21}\right) \\ &= \frac{125}{21}. \end{aligned}$$

We can then find w by remembering $w = v + 3a$.

$$\begin{aligned} w &= v + 3a \\ &= \frac{125}{21} + 3\left(\frac{10}{21}\right) \\ &= \frac{155}{21}. \end{aligned}$$

We ultimately want t_3.

$$\begin{aligned} s_3 &= u_3 t_3 + \frac{1}{2} a_3 t_3^2 \\ \Rightarrow 20 &= \frac{155}{21} t_3 + \frac{10}{21} t_3^2 \\ \Rightarrow 420 &= 155 t_3 + 10 t_3^2 \\ \Rightarrow 84 &= 33 t_3 + 2 t_3^2 \\ \Rightarrow 0 &= 2 t_3^2 + 33 t_3 - 84 \\ \Rightarrow t_3 &= 2.24, \cancel{-18.74} \text{ seconds.} \end{aligned}$$

using the quadratic formula.

Method 2: We continue identically as far as replacing the extra equations with algebra.

First Part	Second Part	Third Part	Extra Equations
$u_1 =$	$u_2 = v$	$u_3 = w$	$\cancel{u_2 = v_1}$
$v_1 = v$	$v_2 = w$	$v_3 =$	$\cancel{u_3 = v_2}$
$a_1 = a$	$a_2 = a$	$a_3 = a$	$\cancel{a_1 = a_2 = a_3}$
$s_1 = 20$	$s_2 = 20$	$s_3 = 20$	
$t_1 = 4$	$t_2 = 3$	$t_3 =$	

Then

$$\begin{aligned} s_1 &= v_1 t_1 - \frac{1}{2} a_1 t_1^2 \\ \Rightarrow 20 &= 4v - 8a \\ \Rightarrow 20 + 8a &= 4v \\ \Rightarrow 5 + 2a &= v. \end{aligned}$$

We now “know” v, at least in terms of the other variables. If we wanted to we could update our UVAST array accordingly.

First Part	Second Part	Third Part	Extra Equations
$u_1 =$	$u_2 = \cancel{v}\ 5 + 2a$	$u_3 = w$	$\cancel{u_2 = v_1}$
$v_1 = \cancel{v}\ 5 + 2a$	$v_2 = w$	$v_3 =$	$\cancel{u_3 = v_2}$
$a_1 = a$	$a_2 = a$	$a_3 = a$	$\cancel{a_1 = a_2 = a_3}$
$s_1 = 20$	$s_2 = 20$	$s_3 = 20$	
$t_1 = 4$	$t_2 = 3$	$t_3 =$	

Note that our UVAST array now only has two variables. Moving on to the second column,

$$\begin{aligned} v_2 &= u_2 + a_2 t_2 \\ \Rightarrow w &= 5 + 2a + 3a \\ &= 5 + 5a. \end{aligned}$$

We now "know" w, at least in terms of a. If we wanted to we could update our UVAST array accordingly.

First Part	Second Part	Third Part	Extra Equations
$u_1 =$	$u_2 = \cancel{v}\ 5+2a$	$u_3 = \cancel{w}\ 5+5a$	~~$u_2 = v_1$~~
$v_1 = \cancel{v}\ 5+2a$	$v_2 = \cancel{w}\ 5+5a$	$v_3 =$	~~$u_3 = v_2$~~
$a_1 = a$	$a_2 = a$	$a_3 = a$	~~$a_1 = a_2 = a_3$~~
$s_1 = 20$	$s_2 = 20$	$s_3 = 20$	
$t_1 = 4$	$t_2 = 3$	$t_3 =$	

Now our UVAST array only has one variable. Using another equation on the second column to get an equation with only a,

$$\begin{aligned} s_2 &= \left(\frac{u_2 + v_2}{2}\right) t_2 \\ \Rightarrow 20 &= \left(\frac{5 + 2a + 5 + 5a}{2}\right) 3 \\ \Rightarrow \frac{20}{3} &= \frac{10 + 7a}{2} \\ \Rightarrow \frac{40}{3} &= 10 + 7a \\ \Rightarrow \frac{10}{3} &= 7a \\ \Rightarrow \frac{10}{21} &= a. \end{aligned}$$

From her we can find

$$\begin{aligned} w &= 5 + 5a \\ &= 5 + 5\left(\frac{10}{21}\right) \\ &= \frac{155}{21} \end{aligned}$$

and

$$\begin{aligned} v &= 5 + 2a \\ &= 5 + 2\left(\frac{10}{21}\right) \\ &= \frac{125}{21}. \end{aligned}$$

Now that we have found our variables we can then find $t_3 = 2.24$ seconds identically to Method 1.

Note 1.42 In this example, we avoided using the “blank” variables in the UVAST array u_1, v_3 and t_3 in equations. After replacing the extra equations with algebra this is good practice; they are unhelpful for solving for the variables you created. Only find them at the end if they are relevant to the answer.

On a related note, notice again how Step 3 of Rule 1.37 is vindicated. We used one equation on column 1, two on column 2 and none on column 3 when solving for the simultaneous equations.

There is a wide algebraic principle at work in Method 1 here, which extends far beyond Linear Motion of even Applied Maths.

Rule 1.43 When trying to solve for n variables, you need n simultaneous equations.

Note 1.44 In Example 1.39 we needed to solve for a and s and so set up a system of two simultaneous equations. In Example 1.41 we needed to solve for a, v and w and so in Method 1 set up a system of three simultanoues equation. This is a principle that extends far beyond Linear Motion and even Applied Maths.

Question 1.45 An object can accelerate at a rate of 4 m/s^2 and decelerate at a rate of 8 m/s^2. What is the shortest time in which it can travel a distance of 1200 m, starting and ending the journey at rest?

When variables are given in the question they should be placed in the UVAST array as if they were known quantities.

Question 1.46 A cyclist starts from rest with acceleration a for 10 seconds. She then travels at a constant speed for 15 seconds. Her total distance travelled was 500 m. What is a? (**Hint:** place a in the UVAST array).

Question 1.47 A car accelerating uniformly covers 35 m in one second, 37 m in the next second and 39 m in the next second. Find its initial speed. (**Hint:** treat this as a three-part journey where each second is one part, with the same acceleration).

1.6 Velocity-Time Graphs

Now that we’re dealing with multi-part journeys, it can sometimes help to draw a velocity-time graph, that is, a graph with time on the horizontal axis and velocity on the vertical axis (drawing a time-velocity graph was also part of Q1 on the Leaving Cert exam, most recently in 2022). Say an object accelerates uniformly from rest to a speed of 40 m/s in 8 seconds. It maintains this speed for 15 seconds, then decelerates uniformly to rest over 10 seconds. Its velocity-time graph can be shown in either of the following ways.

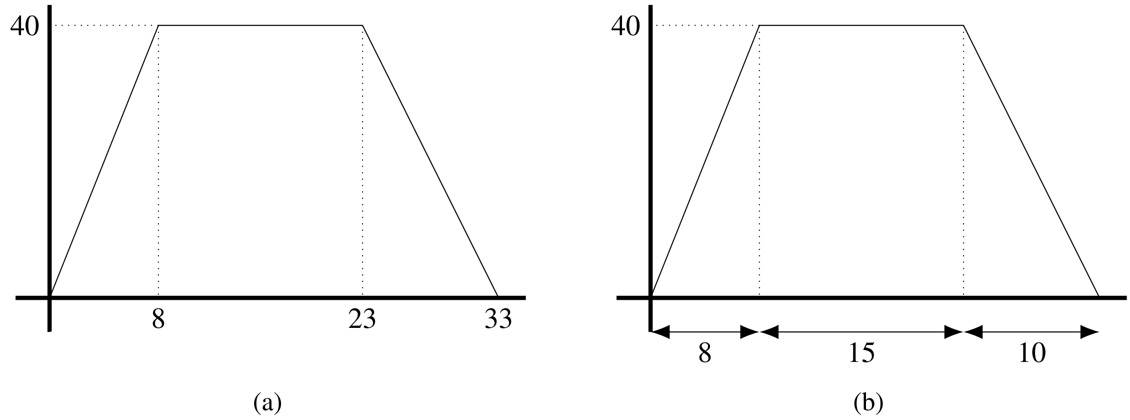


Figure 1.7

Note that regular increments don't need to be placed on the axes, just the relevant values. The relevant values are all heights, and either all times (like in Fig 1.7 (a)) or the length of all time intervals (like in Fig 1.7 (b)). Also note that the slope of the first and last lines are 5 and -4 respectively, which are the accelerations in these parts of the journey. As we are only dealing with constant acceleration, each piece of our time-velocity graph will be linear.

Note 1.48 Velocity-time graphs are sometimes called time-velocity graphs, or speed-time graphs. As you will never be asked to draw a time-velocity graph when velocity is negative, there is no distinction between velocity or speed.

One added benefit of time-velocity graphs is that the area under the graph is equal to the total distance travelled.

Rule 1.49 Given the time-velocity graph of an object, its total distance travelled is equal to the area between the graph and the x-axis (i.e. the area under the graph).

For example, using formulae for areas of triangles and rectangles we can see that the area under the curve in Fig 1.7 is equal to

$$\begin{aligned}\frac{1}{2}(8)(40)+15(40)+\frac{1}{2}(10)(40) &= 160+600+200 \\ &= 960.\end{aligned}$$

If you notice, the velocity-time graph in Figure 1.7 is exactly that of the car in Question 1.32, and 960 m is the answer we got in Question 1.32 (b).

Question 1.50 A cyclist starts from rest and accelerates uniformly to a velocity of 10 m/s in 5 seconds. She continues at this speed for 6 seconds, and then decelerates uniformly to rest in 8 seconds.

(a) Draw a time-velocity graph of the journey.

(b) Calculate the total area under the graph.

Note 1.51 In all examples so far we could draw a velocity-time graph with all relevant points on the graph numbered. One can always draw a rough time-velocity graph, but may not have all the numbers depending on the initial information in the question. If drawing a time-velocity graph for your own benefit you can leave certain values blank. If asked to draw a time-velocity graph in exam questions it is usually the first part of the question and so it is expected that some values may be left blank. We'll talk more about this later (see the exam question in Example 1.83).

Example 1.52 A cyclist accelerates from rest for 5 seconds. She then decelerates until she reaches a speed of 6 m/s. Draw a time-velocity graph for the journey of the car (without doing any calculations).

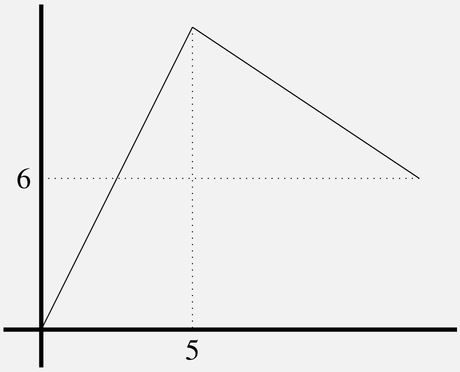


Figure 1.8

See that we can't give a value for the maximum speed on the y-axis, or the time at the end of the second part of the journey.

Question 1.53 A car passes a point P at 12 m/s, accelerating until it reaches a speed of 30 m/s. It continues at this speed for 12 seconds and then slows to rest at a point Q in 5 seconds. Draw a time-velocity graph for the journey of the car.

1.7 Multi-Object Journeys

We will now cover situations involving more than one object. In problems like these each part of the journey of each object leads to a column in the UVAST array. There is also usually a connection between the total time and displacement of the objects, leading to more extra equations. We will still solve these problems by applying Rule 1.37.

Example 1.54 A car starts from rest at a traffic light and accelerates at a rate of 3 m/s^2. At the same time it is passed by a bike travelling at a constant speed of of 12 m/s. After how long will the car overtake the bike?

Assuming the journey for each vehicle starts as they pass the traffic lights and end when they meet, our UVAST array can be written as follows.

Car	Bike	Extra Equations
$u_C = 0$	$u_B = 12$	$s_C = s_B$
$v_C =$	$v_B = 12$	$t_C = t_B$
$a_C = 3$	$a_B = 0$	
$s_C =$	$s_B =$	
$t_C =$	$t_B =$	

Let $s_C, s_B = s$ and $t_C, t_B = t$ so that our UVAST array becomes

Car	Bike	Extra Equations
$u_C = 0$	$u_B = 12$	~~$s_C = s_B$~~
$v_C =$	$v_B = 12$	~~$t_C = t_B$~~
$a_C = 3$	$a_B = 0$	
$s_C = s$	$s_B = s$	
$t_C = t$	$t_B = t$	

Then

$$s_C = u_C t_C + \frac{1}{2} a_C t_C^2$$
$$\Rightarrow s = \frac{3}{2} t^2.$$

Also

$$s_B = u_B t_B + \frac{1}{2} a_B t_B^2$$
$$\Rightarrow s = 12t.$$

We then have a system of two simultaneous equations in two variables.

$$s = \frac{3}{2} t^2,$$
$$s = 12t.$$

Letting the expressions for s be equal,

$$12t = \frac{3}{2} t^2$$
$$\Rightarrow 0, 8 \text{ seconds} = t.$$

Ignoring $t = 0$, our answer is that the car and bike will meet after 8 seconds.

Question 1.55 A car starts from rest with an acceleration of 3 m/s^2. 10 seconds later a motorbike passes the same point at a speed of 10 m/s and an acceleration of 5 m/s^2. How long after the car starts its journey are both vehicles at the same speed? After how long do they meet?

What is interesting about Question 1.55 is that the car and motorbike are not travelling for the same amount of time. In other questions they may not travel the same distance, or even travel in the same direction.

Example 1.56 Two points, P and Q, lie on a straight level road 1.18 km apart. A car passes point P in the direction of Q at a speed of 5 m/s and with an acceleration of 2 m/s^2. At the same time, a motorbike passes point Q in the direction of P at a speed of 4 m/s and with an acceleration of 3 m/s^2. How far from point P will the car and motorbike meet?

Here we could have the following UVAST array.

Car	Motorbike	Extra Equations
$u_C = 5$	$u_M = 4$	$s_C + s_M = 1180$
$v_C =$	$v_M =$	$t_C = t_M$
$a_C = 2$	$a_M = 3$	
$s_C =$	$s_M =$	
$t_C =$	$t_M =$	

If we imagine the line PQ as being from left to right we are letting the rightward direction be positive for the car, but the leftward direction be positive for the motorbike. This doesn't lead to any contradictions. We are also imagining the journeys to start when the car passes P and the motorbike passes Q, and end when they meet each other, which is why $t_C = t_M$, but $s_C + s_T = 1180$.

Let $t_C = t_M = t$, and $s_C = s$ so that our UVAST array becomes

Car	Motorbike	Extra Equations
$u_C = 5$	$u_M = 4$	~~$s_C + s_M = 1180$~~
$v_C =$	$v_M =$	~~$t_C = t_M$~~
$a_C = 2$	$a_M = 3$	
$s_C = s$	$s_M = 1180 - s$	
$t_C = t$	$t_M = t$	

There is no reason we chose to let $s_C = s$ rather than s_M, except perhaps that the answer to the question is s_C, the distance from P to where the car and motorbike meet. From the car column,

$$\begin{aligned} s_C &= u_C t_C + \frac{1}{2} a_C t_C^2 \\ \Rightarrow s &= 5t + t^2. \end{aligned}$$

From the motorbike column,

$$\begin{aligned} s_M &= u_M t_M + \frac{1}{2} a_M t_M^2 \\ \Rightarrow 1180 - s &= 4t + \frac{3}{2} t^2 \\ \Rightarrow 1180 - 4t - \frac{3}{2} t^2 &= s. \end{aligned}$$

Setting these expressions for s equal we have

$$\begin{aligned} 5t + t^2 &= 1180 - 4t - \frac{3}{2} t^2 \\ \Rightarrow \frac{5}{2} t^2 + 9t - 1180 &= 0 \\ \Rightarrow t &= -23.6,\ 20. \end{aligned}$$

Ignoring the negative answer we get $t = 20$ seconds, so that our answer is

$$\begin{aligned} s &= 5(20) + 20^2 \\ &= 500 \text{ m}. \end{aligned}$$

Note 1.57 Notice how, if we let $s_M = s$ instead, we wouldn't be done after finding s (but in the stress of an exam we might think we are)! Sometimes there is a reason to choose one subscripted variable to be equal to the letter variable rather than it being completely arbitrary.

In Rule 1.37 we only considered extra equations with two subscript variables. The trick with converting extra equations with three or more variables to algebra is that it is almost guaranteed that some of them can be calculated immediately.

Rule 1.58 When applying Rule 1.37 to solve Linear Motion problems, if an extra equation has more than two subcripted variables it is almost certain that all but two of them can be calculated directly from the information in the question. In the rare situation where this is not the case, let all but one subscripted variable equal a lettered variable and write the last subscripted variable in terms of all of the lettered variables.

Example 1.59 A car starts from rest to a speed of 24 m/s over 8 seconds, and then continues at this speed. At the same time, a truck starts from rest at the same point with an acceleration of 2 m/s^2. After how long do the truck and car meet? Ignore the length of the car and truck.

In this question, we are specifically looking for the time that the displacement of both vehicles are equal. This won't happen before the car stops accelerating (as its acceleration is greater than that of the truck), so we know that the car completes the first part of its journey. Letting s_{C1} and s_{C2}, etc. be the variables describing the first and second part of the car's journey, and s_T, etc. be those of the truck, we have

Car Part 1	Car Part 2	Truck	Extra Equations
$u_{C1} = 0$	$u_{C2} = 24$	$u_T = 0$	$s_{C1} + s_{C2} = s_T$
$v_{C1} = 24$	$v_{C2} = 24$	$v_T =$	$8 + t_{C2} = t_T$
$a_{C1} =$	$a_{C2} = 0$	$a_T = 2$	
$s_{C1} =$	$s_{C2} =$	$s_T =$	
$t_{C1} = 8$	$t_{C2} =$	$t_T =$	

By our extra equation that we are assuming that the journey ends when the truck and car are at the same position, so that their total displacement is the same. Notice also the connection between the car times and truck times; the total amount of time for the whole journey is the same for both vehicles.

See that s_{C1} can be calculated straight away.

$$\begin{aligned} s_{C1} &= \left(\frac{u_{C1} + v_{C2}}{2}\right) t_{C2} \\ &= \left(\frac{u_{C1} + v_{C2}}{2}\right) t_{C2} \\ &= \left(\frac{0 + 24}{2}\right) 8 \\ &= 96. \end{aligned}$$

Therefore the extra equation $s_{C1} + s_{C2} = s_T$ can be simplified to $96 + s_{C2} = s_T$ which only has two variables. Then letting $s_T = s$, $s_{C2} = s - 96$, and letting $t_T = t$, $s_{C2} = t - 8$ so that our UVAST array becomes

Car Part 1	Car Part 2	Truck	Extra Equations
$u_{C1} = 0$	$u_{C2} = 24$	$u_T = 0$	~~$s_{C1} + s_{C2} = s_T$~~
$v_{C1} = 24$	$v_{C2} = 24$	$v_T =$	~~$8 + t_{C2} = t_T$~~
$a_{C1} =$	$a_{C2} = 0$	$a_T = 2$	
$s_{C1} = 96$	$s_{C2} = s - 96$	$s_T = s$	
$t_{C1} = 8$	$t_{C2} = t - 8$	$t_T = t$	

From the second column,

$$\begin{aligned} s_{C2} &= u_{C2}t_{C2} + \frac{1}{2}a_{C2}t_{C2}^2 \\ \Rightarrow s - 96 &= 24(t-8) \\ \Rightarrow s &= 24t - 96. \end{aligned}$$

From the third column,

$$\begin{aligned} s_T &= u_T t_T + \frac{1}{2}a_T t_T^2 \\ \Rightarrow s &= t^2. \end{aligned}$$

Letting these expressions for s be equal gives us

$$\begin{aligned} t^2 &= 24t - 96 \\ \Rightarrow t &= 5.07, 18.93 \text{ seconds.} \end{aligned}$$

Our equations don't make sense for $t < 8$, so we discard $t = 5.07$ to get that the car and truck meet after 18.93 seconds.

Note 1.60 As we have seen a few times now, we often solve a quadratic to get two answers when we only expect one. Sometimes the question implies that there is only one by asking for "the value of..." rather than "the possible values of...". In that case if you get two answers there is usually a reason to discard one. This advice extends beyond Applied Maths.

Question 1.61 Train S starts at rest from station A in the direction of station B. It accelerates uniformly at 3 m/s^2 to a speed of 30 m/s, and then continues at this speed. At the same time Train T starts at rest from station B in the direction of station A. It accelerates uniformly at 2 m/s^2 to a speed of 50 m/s, and then continues at this speed. Stations A and B are 3.225 km apart. How long will it take for the trains to meet each other?

Question 1.62 A car leaves a set of traffic lights, starting from rest, with an acceleration of 2 m/s^2 until it reaches a speed of 20 m/s. At the same time a motorbike leaves the same traffic lights, also starting from rest, with an acceleration of 3 m/s^2 until it reaches a speed of 15 m/s. Given that they meet after the car stops accelerating, after how many seconds to the car and bike meet?

Question 1.63 A train can accelerate at a rate of 1 m/s^2 and decelerate at a rate of 2 m/s^2. It has a maximum speed of 20 m/s. How quickly can it get from station A to station B if they are 1 km apart and it must start and end the journey at rest?

Let's consider the "rare case" referred to in Rule 1.58.

Example 1.64 A car accelerates uniformly from rest with acceleration a until it reaches a speed of 10 m/s. It continues at this speed for some time, before decelerating to rest at a rate of $2a$. The total time of the journey is 60 seconds and the car covers 500 m. What is a?

We have the following UVAST array.

Car Part 1	Car Part 2	Truck	Extra Equations
$u_1 = 0$	$u_2 = 10$	$u_3 = 10$	$t_1 + t_2 + t_3 = 60$
$v_1 = 10$	$v_2 = 10$	$v_3 = 0$	$s_1 + s_2 + s_3 = 500$
$a_1 = a$	$a_2 = 0$	$a_3 = -2a$	
$s_1 =$	$s_2 =$	$s_3 =$	
$t_1 =$	$t_2 =$	$t_3 =$	

We can't figure out t_1, t_2 or t_3, so letting $t_1 = t$, $t_3 = T$, our extra equation gives us

$$\begin{aligned} t_2 &= 60 - t_1 - t_3 \\ &= 60 - t - T. \end{aligned}$$

Similarly, let $s_1 = s$, $s_3 = S$ and $s_2 = 500 - s - S$ so that we can write our UVAST array as

Part 1	Part 2	Part 3	Extra Equations
$u_1 = 0$	$u_2 = 10$	$u_3 = 10$	~~$t_1 + t_2 + t_3 = 60$~~
$v_1 = 10$	$v_2 = 10$	$v_3 = 0$	~~$s_1 + s_2 + s_3 = 500$~~
$a_1 = a$	$a_2 = 0$	$a_3 = -2a$	
$s_1 = s$	$s_2 = 500 - s - S$	$s_3 = S$	
$t_1 = t$	$t_2 = 60 - t - T$	$t_3 = T$	

We have five variables and must therefore make five variables to solve using simultaneous equations. Instead let's substitute out the variables one by one. Using the first column,

$$\begin{aligned} v_1 &= u_1 + a_1 t_1 \\ \Rightarrow 10 &= at \\ \Rightarrow \frac{10}{a} &= t. \end{aligned}$$

Again using the first column,

$$\begin{aligned} s_1 &= \left(\frac{u_1 + v_1}{2}\right) t_1 \\ \Rightarrow s &= (5)\frac{10}{a} \\ &= \frac{50}{a}. \end{aligned}$$

Using the third column,

$$\begin{aligned} v_3 &= u_3 + a_3 t_3 \\ \Rightarrow 0 &= 10 - 2aT \\ \Rightarrow T &= \frac{5}{a}. \end{aligned}$$

Again using the third column,

$$\begin{aligned} s_3 &= \left(\frac{u_3 + v_3}{2}\right) t_3 \\ \Rightarrow S &= (5)\frac{5}{a} \\ &= \frac{25}{a}. \end{aligned}$$

At this point we might revisit our UVAST array, which now looks like this.

Part 1	Part 2	Part 3	Extra Equations
$u_1 = 0$	$u_2 = 10$	$u_3 = 10$	~~$t_1 + t_2 + t_3 = 60$~~
$v_1 = 10$	$v_2 = 10$	$v_3 = 0$	~~$s_1 + s_2 + s_3 = 500$~~
$a_1 = a$	$a_2 = 0$	$a_3 = -2a$	
$s_1 = \frac{50}{a}$	$s_2 = 500 - \frac{75}{a}$	$s_3 = \frac{25}{a}$	
$t_1 = \frac{10}{a}$	$t_2 = 60 - \frac{15}{a}$	$t_3 = \frac{5}{a}$	

Now using the second column,

$$\begin{aligned} s_2 &= u_2 t_2 \\ \Rightarrow 500 - \frac{75}{a} &= 10\left(60 - \frac{15}{a}\right) \\ &= 600 - \frac{150}{a} \\ \Rightarrow \frac{150}{a} - \frac{75}{a} &= 600 - 500 \\ \Rightarrow \frac{75}{a} &= 100 \\ \Rightarrow 75 &= 100a \\ \Rightarrow 0.75 \text{ m/s}^2 &= a. \end{aligned}$$

Note 1.65 This example is arguably too contrived to be considered a likely exam question, but for a student trying to cover all of their bases it serves as an example when we must properly consider extra equations with more than two variables.

Question 1.66 A car accelerates uniformly from rest with acceleration 2 m/s^2 until it reaches its maximum speed. It continues at this speed for some time, before decelerating to rest at a rate of 4 m/s^2. The total time of the journey is 40 seconds and the car covers 1000 m. What is its maximum speed?

1.8 More Algebraic Problems

Note 1.67 At this point we have covered enough material to answer about 70% of Leaving Cert exam questions on Linear Motion. This section deals with the most abstruse questions.

We may have problems where the solution is not a number, but instead a proof of an algebraic equation. In this case we apply the following variation of Rule 1.37.

Rule 1.68 If trying to prove an equality for variables given in a question, apply the following steps.

1. Create the UVAST array as normal, using any variables and numbers given in the question.
2. Replace the extra equations with variables as normal.
3. Consider any variables in the equation you're trying to prove as **equation variables**, and any others in the UVAST array as **non-equation variables**. By either setting up simultaneous equations or finding each non-equation variable one by one in terms of the other remaining (equation or non-equation) variables, remove the non-equation variables from the UVAST array until only equation variables are left.
4. Use an equation of motion and the UVAST array to write an equation in only the equation variables. Subject to rearranging this should be the equation you are being asked to prove.

When creating equations in steps 3-4 using the equations of motion on each column, obey the following rules.

- Don't apply the same equation of motion more than once to the same column.
- If a column is filled entirely with variables, numbers and $a \neq 0$, you can apply two equations to the same column. It doesn't matter what the second one is as long as it is not the same as the first one.
- The number of equations of motion you can apply to a column is at most 2. Subtract 1 for each empty subscripted variable in the column, and subtract another 1 if $a = 0$.

Example 1.69 A racecar is at rest on the track. There is a technician s metres straight ahead; he will stop the clock when the racecar stops beside him. The racecar has maximum acceleration a and maximum deceleration d. It the racecar accelerates and then immediately decelerates at its maximum rates so that it stops just at the technician, show that the total time of the journey, $t = \sqrt{\frac{(2s)(a+d)}{ad}}$.

---

1. We will first write an initial UVAST array that contains s, t, a, d and extra equations.

First Part	Second Part	Extra Equations
$u_1 = 0$	$u_2 =$	$v_1 = u_2$
$v_1 =$	$v_2 = 0$	$s_1 + s_2 = s$
$a_1 = a$	$a_2 = -d$	$t_1 + t_2 = t$
$s_1 =$	$s_2 =$	
$t_1 =$	$t_2 =$	

2. Letting

$$v_1 = V,$$
$$s_1 = S,$$
$$t_1 = T,$$

we can write the UVAST array as

First Part	Second Part	Extra Equations
$u_1 = 0$	$u_2 = V$	~~$v_1 = u_2$~~
$v_1 = V$	$v_2 = 0$	~~$s_1 + s_2 = s$~~
$a_1 = a$	$a_2 = -d$	~~$t_1 + t_2 = t$~~
$s_1 = S$	$s_2 = s - S$	
$t_1 = T$	$t_2 = t - T$	

Notice by using capital letters we can easily distinguish between our non-equation and equation variables.

3. Using the first column,

$$v_1 = u_1 + a_1 t_1$$
$$\Rightarrow V = aT.$$

We have now written V in terms of other variables, and can consider it "removed" from the UVAST array, which now looks like this if you want to update it.

First Part	Second Part	Extra Equations
$u_1 = 0$	$u_2 = \cancel{V}\ aT$	~~$v_1 = u_2$~~
$v_1 = \cancel{V}\ aT$	$v_2 = 0$	~~$s_1 + s_2 = s$~~
$a_1 = a$	$a_2 = -d$	~~$t_1 + t_2 = t$~~
$s_1 = S$	$s_2 = s - S$	
$t_1 = T$	$t_2 = t - T$	

Moving on to the second column,

$$v_2 = u_2 + a_2 t_2$$
$$\Rightarrow 0 = aT - d(t - T)$$
$$\Rightarrow 0 = aT - dt + dT$$
$$\Rightarrow dt = (a + d)T$$
$$\Rightarrow \frac{dt}{a+d} = T.$$

Again, T is now removed from our UVAST array which we can update if we so choose.

First Part	Second Part	Extra Equations
$u_1 = 0$	$u_2 = \cancel{V}\ \cancel{aT}\ \frac{adt}{a+d}$	~~$v_1 = u_2$~~
$v_1 = \cancel{V}\ \cancel{aT}\ \frac{adt}{a+d}$	$v_2 = 0$	~~$s_1 + s_2 = s$~~
$a_1 = a$	$a_2 = -d$	~~$t_1 + t_2 = t$~~
$s_1 = S$	$s_2 = s - S$	
$t_1 = \cancel{T}\ \frac{dt}{a+d}$	$t_2 = \cancel{t - T}\ t - \frac{dt}{a+d}$	

S is the only non-equation variable we still have to remove. Back to the first column, which we can use a second equation on (and it doesn't matter which one)

$$\begin{aligned} s_1 &= u_1 t_1 + \frac{1}{2} a_1 t_1^2 \\ \Rightarrow S &= \frac{1}{2} a \left(\frac{dt}{a+d} \right)^2 \\ &= \frac{a}{2} \frac{d^2 t^2}{(a+d)^2} \\ &= \frac{a d^2 t^2}{2(a+d)^2}. \end{aligned}$$

Now our UVAST has only equation variables.

<u>First Part</u>	<u>Second Part</u>	<u>Extra Equations</u>
$u_1 = 0$	$u_2 = \cancel{V}\ \cancel{aT}\ \frac{adt}{a+d}$	$\cancel{v_1 = u_2}$
$v_1 = \cancel{V}\ \cancel{aT}\ \frac{adt}{a+d}$	$v_2 = 0$	$\cancel{s_1 + s_2 = s}$
$a_1 = a$	$a_2 = -d$	$\cancel{t_1 + t_2 = t}$
$s_1 = \cancel{S}\ \frac{ad^2t^2}{2(a+d)^2}$	$s_2 = \cancel{s - S}\ s - \frac{ad^2t^2}{2(a+d)^2}$	
$t_1 = \cancel{T}\ \frac{dt}{a+d}$	$t_2 = \cancel{t - T}\ t - \frac{dt}{a+d}$	

4. We have already used two equations on the first column, so we have to go to the second column.

$$\begin{aligned} s_2 &= u_2 t_2 + \frac{1}{2} a_2 t_2^2 \\ \Rightarrow s - \frac{ad^2t^2}{2(a+d)^2} &= \frac{adt}{a+d}\left(t - \frac{dt}{a+d}\right) + \frac{1}{2}(-d)\left(t - \frac{dt}{a+d}\right)^2 \end{aligned}$$

Subject to rearranging, this equation should be the one we are asked to prove.

$$\begin{aligned} s - \frac{ad^2t^2}{2(a+d)^2} &= \frac{adt}{a+d}\left(t - \frac{dt}{a+d}\right) + \frac{1}{2}(-d)\left(t - \frac{dt}{a+d}\right)^2 \\ &= \frac{adt^2}{a+d} - \frac{ad^2t^2}{(a+d)^2} - \frac{d}{2}\left(\frac{t(a+d)}{a+d} - \frac{dt}{a+d}\right)^2 \\ &= \frac{adt^2}{a+d} - \frac{ad^2t^2}{(a+d)^2} - \frac{d}{2}\left(\frac{ta+td}{a+d} - \frac{dt}{a+d}\right)^2 \\ &= \frac{adt^2}{a+d} - \frac{ad^2t^2}{(a+d)^2} - \frac{d}{2}\left(\frac{ta}{a+d}\right)^2 \\ &= \frac{adt^2}{a+d} - \frac{ad^2t^2}{(a+d)^2} - \frac{d}{2}\frac{a^2t^2}{(a+d)^2} \\ &= \frac{adt^2}{a+d} - \frac{ad^2t^2}{(a+d)^2} - \frac{a^2dt^2}{2(a+d)^2} \\ \Rightarrow s &= \frac{adt^2}{a+d} - \frac{ad^2t^2}{(a+d)^2} - \frac{a^2dt^2}{2(a+d)^2} + \frac{ad^2t^2}{2(a+d)^2} \\ &= \frac{adt^2}{a+d} - \frac{ad^2t^2}{2(a+d)^2} - \frac{a^2dt^2}{2(a+d)^2}. \end{aligned}$$

Adding the fraction,

$$\begin{aligned} s &= \frac{2(a+d)adt^2}{2(a+d)^2} - \frac{ad^2t^2 + a^2dt^2}{2(a+d)^2} \\ &= \frac{2a^2dt^2 + 2ad^2t^2 - ad^2t^2 - a^2dt^2}{2(a+d)^2} \\ &= \frac{a^2dt^2 + ad^2t^2}{2(a+d)^2} \\ &= \frac{adt^2(a+d)}{2(a+d)^2} \\ s &= \frac{adt^2}{2(a+d)}. \end{aligned}$$

Re-writing t as the subject,

$$\begin{aligned} s &= \frac{adt^2}{2(a+d)} \\ \Rightarrow 2s(a+d) &= adt^2 \\ \Rightarrow \frac{2s(a+d)}{ad} &= t^2 \\ \Rightarrow \sqrt{\frac{2s(a+d)}{ad}} &= t, \end{aligned}$$

as required.

Note 1.70 Notice the philosophy near the end of our calculations. Although we were asked to show that

$$t = \sqrt{\frac{2s(a+d)}{ad}},$$

our intermediate goal was just to get s, t, a and d in the same equation with no other variables. Our final goal was then to write our equation in the form $t = \ldots$. If we could do all of this without making any mistakes then, for subtle mathematical reasons that we won't get into here, our equation had to be the one asked for in the question (barring some minor simplifications on the other side of the equals sign).

Question 1.71 A train has to travel a distance s from rest at one station to rest at the next station. It has an acceleration of a and a deceleration of d. If the train makes the journey in the shortest time possible without having to travel at constant speed, show that its maximum speed v satisfies

$$v = \sqrt{\frac{2ads}{a+d}}.$$

Question 1.72 A car, starting from rest and accelerating uniformly, covers a distance d in the first n seconds of motion and a distance k in the next n seconds. Show that $k = 3d$.

Question 1.73 A train accelerates uniformly from rest to a speed of v m/s. It continues at this speed for a period of time and then decelerates to rest. In travelling a total of d metres the train accelerates through a distance pd and decelerates through a distance of qd. Show that the average speed of the train for the entire journey is

$$\frac{v}{p+q+1}.$$

1.9 Displacement-Time Graphs

Displacement-time graphs are similar to velocity-time graphs except that the y-axis is the displacement of the object rather than the velocity. Unlike velocity-time graphs, it is important to understand that the graph is showing the displacement of an object from some reference point, not the distance it has travelled.

Note 1.74 There are three important properties to note when looking at a displacement-time graphs at a specific point:

- Is the graph positive or negative? If it is positive then the displacement of the object is in the positive direction from the reference point at this time. If it is negative then the displacement of the object is in the negative direction. If it is 0, i.e. on the horizontal axis then the object is at the reference point.
- Is the graph increasing or decreasing? If it is increasing then the velocity of the object is positive at this time, if it is decreasing it is negative. If it is neither then the object is at rest. The slope of the tangent line to the curve at a point is its velocity at that point.
- Is the graph increasing by more or less as you look from left to right? If it is increasing by more then acceleration is positive, if it is increasing by less then acceleration is negative. If it is neither (i.e. the graph is linear) then acceleration is 0, i.e. velocity is constant. By "increasing more", we mean this in the inequalities sense, not in the sense of magnitudes. Therefore "decreasing, but decreasing less" is the same as "increasing more", much like how $-1 > -2$ but $1 < 2$. Graphs that are increasing faster can be described as $\cup$-shaped, or curving anti-clockwise, while graphs that are increasing slower can be described as $\cap$-shaped, or curving clockwise.

Note that each of these properties relate, in turn, to displacement, velocity and acceleration. Although unnecessary to understand this chapter, students who have already studied calculus may recognise that in each bullet point we are considering the original function, its first derivative, and its second derivative (remember that velocity is the derivative of displacement and acceleration is the derivative of velocity).

Note 1.75 There are two ways of remembering the final bullet point. To remember what clockwise and anti-clockwise mean, think "when the object is **A**ccelerating the graph is **A**nti-clockwise". Alternatively, accelerating starts with "a", a vowel, like "u". Decelerating starts with "d", a consonant, like "n". So the graph is **A**ccelerating if it is $\cup$-shaped and **D**ecelerating if it is $\cap$-shaped.

Consider the following six graphs.

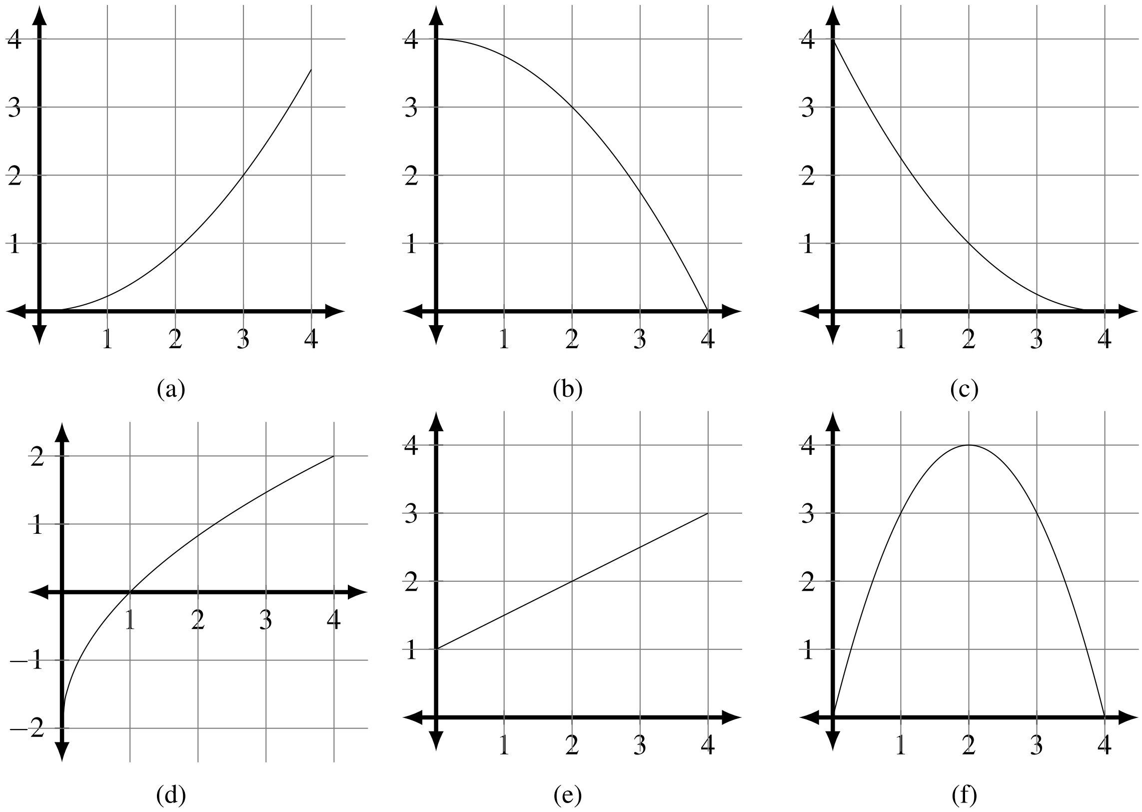


Figure 1.9

Regarding displacement, Figure 1.9 (d) is the only graph with negative displacement. Whatever the reference point is for this journey, the object passes it at time $t = 1$.

Regarding velocity, Figures 1.9 (a), 1.9 (d) and 1.9 (e) have increasing displacement, meaning that their velocity is positive (except at time $t = 0$ in Figure 1.9 (a)). Figures 1.9 (b) and 1.9 (c) have decreasing displacement (except for time $t = 4$ in Figure 1.9 (c)). Figure 1.9 (f) has increasing displacement up to time $t = 2$. For time $t > 2$ it has decreasing displacement. This means that it has positive velocity from time $0 < t < 2$, is at rest at time $t = 2$, and has negative velocity for time $t > 2$.

Regarding acceleration, looking at Figure 1.9 (a), the graph is increasing faster as we move from left to right. If we imagine drawing a tangent line at certain points, the further we go to the right the greater the slope would be. This means that the velocity is increasing (remember that the velocity is equal to the slope) and so acceleration is positive. It can also be described as ∪-shaped, or curving anti-clockwise. The same is true of Figure 1.9 (c); although the graph is decreasing, the slope of the tangent lines are changing from large negative numbers to smaller ones as we move from left to right. Therefore the acceleration is positive. Figures 1.9 (b), 1.9 (d) and 1.9 (f) have negative acceleration by the same logic. Figure 1.9 (e) has zero acceleration as its increase is constant. This is true of all linear displacement-time functions.

Once we have studied the three properties of these graphs, we can build up a story of the journey. For example, in Figure 1.9 (a) the object starts at the reference point. It is at rest as its velocity is 0. It speeds up until at time $t = 4$ is is around 3.5 units away from the reference point. Regarding Figure 1.9 (b), the object starts 4 units away from the reference point, also at rest. It moves closer to the reference point with increasing speed (equivalently decreasing negative velocity) until it reaches the reference point at time $t = 4$.

Question 1.76 Describe the journeys graphed in Figures 1.9 (c)-1.9 (f).

Note 1.77 Question 1.76 is too vague to be considered a Leaving Cert exam question, but students able to answer questions like it show enough of an understanding of displacement-time graphs to be able to answer any question they may be asked.

Question 1.78 For each of the following displacement-time graphs, state the sign of the displacement, velocity and acceleration of the object at times $t = 1$ and $t = 3$.

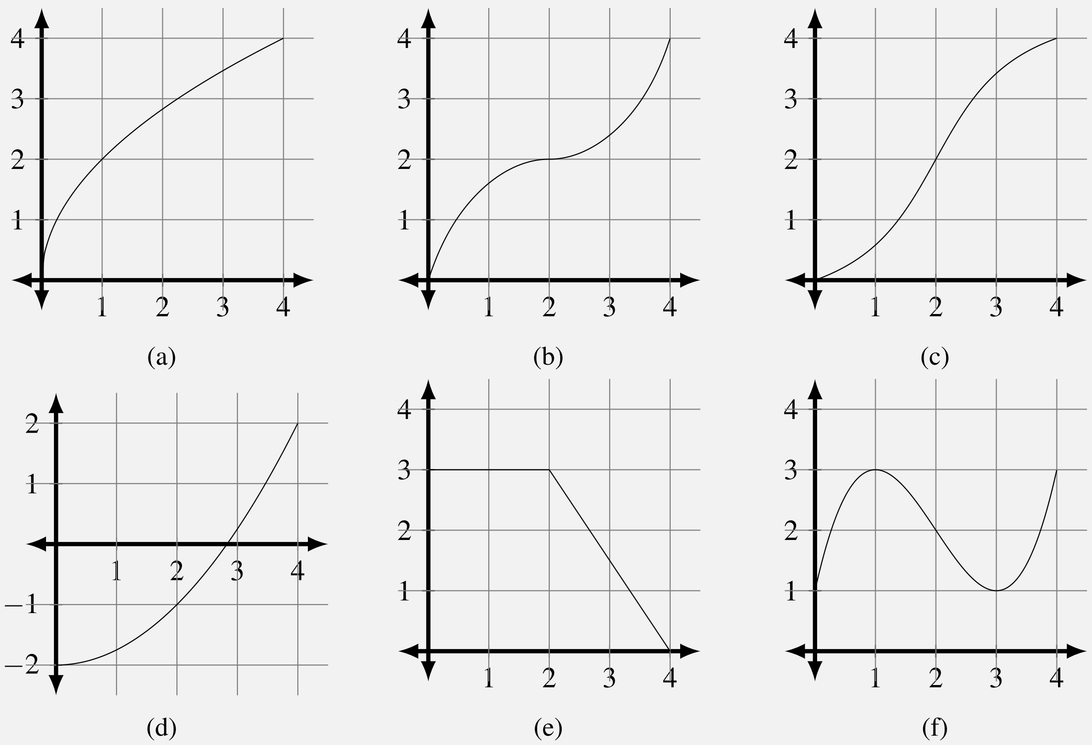


Figure 1.10

Question 1.79 Describe the journeys graphed in Figure 1.10.

Note 1.80 There are some key differences in how we describe journeys and quantities when considering displacement-time graphs.

First, we assume we can have non-zero displacement at time $t = 0$. This is in contrast to how we approach Linear Motion calculations. This is because displacement-time graphs show the displacement of an object from a reference point at a given time, whereas in our calculations displacement was always measured from the starting point of the journey.

Second, displacement-time graphs can be understood even when acceleration is non-constant, and all observations made in this section remain true in the case of non-constant acceleration.

1.10 Exam Questions

Example 1.81 — 2021 Q1 (a).

(a) A ball is thrown vertically downwards from the top of a building of height h m. The ball passes the top half of the building in 1.2 s and takes a further 0.8 s to reach the bottom of the building.
Find

(i) the value of h

(ii) the speed of the ball at the bottom of the building.

Figure 1.11

(i) Although the acceleration of the ball is constant we will split up the journey into two parts; from the top of the building to the middle and from the middle to the bottom. Letting $v_1, u_2 = v$ and letting the downward direction be positive our UVAST array looks as follows.

First Part	Second Part
$u_1 =$	$u_2 = v$
$v_1 = v$	$v_2 =$
$a_1 = g$	$a_2 = g$
$s_1 = \frac{h}{2}$	$s_2 = \frac{h}{2}$
$t_1 = 1.2$	$t_2 = 0.8$

Notice that the object was **thrown**, not dropped, so $u_1 \neq 0$.

We have two variables, v and h, so we want to create two simultaneous equations in two variables, one coming from each column as there is a blank subscripted variable in each column. In the first column,

$$\begin{aligned} s_1 &= v_1 t_1 - \frac{1}{2} a_2 t_1^2 \\ \Rightarrow \frac{h}{2} &= 1.2v - \frac{g}{2}(1.2)^2 \\ \Rightarrow \frac{h}{2} &= 1.2v - 0.72g \\ \Rightarrow h &= 2.4v - 1.44g. \end{aligned}$$

In the second column,

$$\begin{aligned} s_2 &= u_2t_2 + \frac{1}{2}a_2t_2^2 \\ \Rightarrow \frac{h}{2} &= 0.8v + \frac{1}{2}g(0.8)^2 \\ \Rightarrow \frac{h}{2} &= 0.8v + 0.32g \\ \Rightarrow h &= 1.6v + 0.64g. \end{aligned}$$

Letting these expressions for h be equal we have

$$\begin{aligned} 2.4v - 1.44g &= 1.6v + 0.64g \\ \Rightarrow 0.8v &= 2.08g \\ \Rightarrow v &= 2.6g \\ \Rightarrow h &= 2.4(2.6g) - 1.44g \\ \Rightarrow h &= 4.8g \text{ m.} \end{aligned}$$

(ii) To get the speed of the ball at the bottom of the building,

$$\begin{aligned} v &= u + at \\ \Rightarrow v_2 &= 2.6g + 0.8g \\ &= 3.4g \text{ m/s.} \end{aligned}$$

Example 1.82 — 2021 Q1 (b).

(b) Car C, moving with uniform acceleration f passes a point P with speed u (> 0). Two seconds later car D, moving in the same direction with uniform acceleration $2f$ passes P with speed $\frac{6}{5}u$. C and D pass a point Q together. The speeds of C and D at Q are 6.5 m s^{-1} and 9 m s^{-1} respectively.

(i) Show that C travels from P to Q in $(\frac{3}{2f} + 5)$ seconds.

(ii) Find the value of f.

Figure 1.12

(i) We could set up our UVAST array as follows.

Car C	Car D	Extra Equations
$u_C = u$	$u_D = \frac{6}{5}u$	$s_C = s_D$
$v_C = 6.5$	$v_D = 9$	$t_C = t_D + 2$
$a_C = f$	$a_D = 2f$	
$s_C =$	$s_D =$	
$t_C =$	$t_D =$	

Letting $s_C = s$, $t_C = t$, our UVAST array can then be written like this.

Car C	Car D	Extra Equations
$u_C = u$	$u_D = \frac{6}{5}u$	~~$s_C = s_D$~~
$v_C = 6.5$	$v_D = 9$	~~$t_C = t_D + 2$~~
$a_C = f$	$a_D = 2f$	
$s_C = s$	$s_D = s$	
$t_C = t$	$t_D = t - 2$	

We are being asked to show that

$$t = \frac{3}{2f} + 5$$

(why is why we let $t_C = t$ and not t_D) and so will solve this problem by applying Rule 1.68 with t and f being the equation variables and u and s being the non-equation variables which we want to remove from the UVAST array. From the first column,

$$\begin{aligned} v_1 &= u_1 + a_1 t_1 \\ \Rightarrow 6.5 &= u + ft \\ \Rightarrow 6.5 - ft &= u. \end{aligned}$$

Also from the first column,

$$\begin{aligned} s_1 &= \left(\frac{u_1 + v_1}{2}\right) t_1 \\ \Rightarrow s &= \left(\frac{6.5 - ft + 6.5}{2}\right) t \\ &= 6.5t - 0.5ft^2. \end{aligned}$$

Our updated UVAST array looks as follows.

Car C	Car D	Extra Equations
$u_C = \not{u}\ 6.5 - ft$	$u_D = \not{\tfrac{6}{8}}\not{u}\ 7.8 - 1.2ft$	~~$s_C = s_D$~~
$v_C = 6.5$	$v_D = 9$	~~$t_C = t_D + 2$~~
$a_C = f$	$a_D = 2f$	
$s_C = \not{s}\ 6.5t - 0.5ft^2$	$s_D = \not{s}\ 6.5t - 0.5ft^2$	
$t_C = t$	$t_D = t - 2$	

By Rule 1.68 using any more equations on the first column won't help, and we want to avoid t^2 if we are trying to solve for t. Therefore we will choose to apply $v = u + at$ to the second column to get

$$\begin{aligned} v_2 &= u_2 + a_2 t_2 \\ \Rightarrow 9 &= 7.8t - 1.2ft + 2f(t-2) \\ &= 7.8 - 1.2ft + 2ft - 4f \\ \Rightarrow 1.2 + 4f &= 0.8ft \\ \Rightarrow \frac{1.2}{0.8f} + \frac{4}{0.8} &= t \\ \Rightarrow \frac{3}{2f} + 5 &= t, \end{aligned}$$

as required. Note again that our intermediate goal was just to get t and f in the same equation. Our final goal was to rewrite it in the form $t = \ldots$. As if by magic, we then get the equation asked for in the question.

(ii) Note that we now have t in terms of f. That means that we have every indexed variable in terms of f, and can rewrite the second column of our UVAST array (the only column we can still use) as

$$\begin{aligned}
&\underline{\text{Car D}}\\
u_D &= 7.8 - 1.2f\left(\frac{3}{2f}+5\right)\\
&= 7.8 - 1.8 - 6f\\
&= 6 - 6f\\
v_D &= 9\\
a_D &= 2f\\
s_D &= 6.5\left(\frac{3}{2f}+5\right) - 0.5f\left(\frac{3}{2f}+5\right)^2\\
&= \frac{9.75}{f} + 32.5 - 0.5f\left(\frac{2.25}{f^2}+\frac{15}{f}+25\right)\\
&= \frac{9.75}{f} + 32.5 - \frac{1.125}{f} - 7.5 - 12.5f\\
&= \frac{8.625}{f} + 25 - 12.5f\\
t_D &= \frac{1.5}{f} + 3.
\end{aligned}$$

Applying a different equation of motion to this column should yield an equation only containing f, which we should be able to solve. Avoiding equations with squares, mostly out of laziness,

$$\begin{aligned}
s &= \left(\frac{u+v}{2}\right)t\\
\Rightarrow \frac{8.625}{f} + 25 - 12.5f &= \left(\frac{6-6f+9}{2}\right)\left(\frac{1.5}{f}+3\right)\\
&= (7.5-3f)\left(\frac{1.5}{f}+3\right)\\
&= \frac{11.25}{f} + 22.5 - 4.5 - 9f\\
\Rightarrow -\frac{2.625}{f} + 7 - 3.5f &= 0\\
\Rightarrow -2.625 + 7f - 3.5f^2 &= 0\\
\Rightarrow f &= 0.5, 1.5
\end{aligned}$$

by applying the quadratic formula.

The question says find the value of f, not the values. So how do we eliminate one of these values? Well $f = \frac{3}{2}$ implies

$$\begin{aligned}
u_D &= 6 - 6\left(\frac{3}{2}\right)\\
&= -3,
\end{aligned}$$

but we were told that $u > 0$, which implies $u_D > 0$. Therefore our answer is $f = \frac{1}{2}$. This is another example of a principle mentioned all the way back in Note 1.60; if you are asked for the **value** of a variable and get two values, then there must be a reason that one of them should be discarded.

Example 1.83 — 2022 Q1 (a).

(a) A train takes 40 minutes to travel from rest at station A to rest at station B. The distance between the stations is 20 km. The train left station A at 10:00. At 10:15 the speed of the train was 32 km h^{-1} and at 10:30 the speed was 48 km h^{-1}.

The speed of 48 km h^{-1} was maintained until the brakes were applied, causing a uniform deceleration which brought the train to rest at B.

During the first and second 15-minute intervals the accelerations were constant.

(i) Draw a speed-time graph of the motion.

(ii) Find the time taken for the first 16 km.

(iii) Find the deceleration of the train.

Figure 1.13

(i) Our speed-time graph looks as follows.

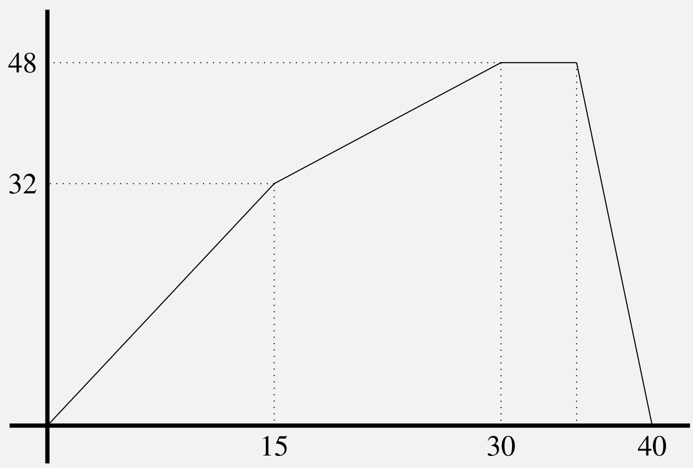


Figure 1.14

We don't know for how long the constant speed of 48 km/hr is maintained, so we don't have a time marked between 30 and 40. We also used minutes on the horizontal axis for the speed-time graph. This is fine as the alternative is awkward decimal numbers for the fractions of hours, but if we used this graph for any mathematics (such as finding the area under the curve) we would need to change the times to hours.

(ii) The question mentions minutes but speed is measured in km/hr. It's easier to convert time to different units than speed, so we will use km and hours in this answer.

Our initial UVAST array is as follows.

First Part	Second Part	Third Part	Fourth Part	Extra Equations
$u_1 = 0$	$u_2 = 32$	$u_3 = 48$	$u_4 = 48$	$t_3 + t_4 = \frac{1}{6}$
$v_1 = 32$	$v_2 = 48$	$v_3 = 48$	$v_4 = 0$	$s_1 + s_2 + s_3 + s_4 = 20$
$a_1 =$	$a_2 =$	$a_3 = 0$	$a_4 =$	
$s_1 =$	$s_2 =$	$s_3 =$	$s_4 =$	
$t_1 = \frac{1}{4}$	$t_2 = \frac{1}{4}$	$t_3 =$	$t_4 =$	

One of our extra equations has four variables, but like in most cases addressed in Rule 1.58 we can find some of the variables immediately before we replace this extra equation with variables.

$$\begin{aligned} s_1 &= \left(\frac{u_1 + v_1}{2}\right) t_1 \\ \Rightarrow s_1 &= \left(\frac{0 + 32}{2}\right) \frac{1}{4} \\ &= 4. \end{aligned}$$

We can similarly find that $s_2 = 10$. Then

$$s_1 + s_2 + s_3 + s_4 = 20$$

becomes

$$\begin{aligned} 4 + 10 + s_3 + s_4 &= 20 \\ \Rightarrow s_3 + s_4 &= 6. \end{aligned}$$

Letting

$$\begin{aligned} t_3 &= t, \\ s_3 &= s, \end{aligned}$$

our UVAST array can then be written as

First Part	Second Part	Third Part	Fourth Part	Extra Equations
$u_1 = 0$	$u_2 = 32$	$u_3 = 48$	$u_4 = 48$	~~$t_3 + t_4 = \frac{1}{6}$~~
$v_1 = 32$	$v_2 = 48$	$v_3 = 48$	$v_4 = 0$	~~$s_1 + s_2 + s_3 + s_4 = 20$~~
$a_1 =$	$a_2 =$	$a_3 = 0$	$a_4 =$	
$s_1 = 4$	$s_2 = 10$	$s_3 = s$	$s_4 = 6 - s$	
$t_1 = \frac{1}{4}$	$t_2 = \frac{1}{4}$	$t_3 = t$	$t_4 = \frac{1}{6} - t$	

Solving for s and t in the usual way using simultaneous equations,

$$\begin{aligned} s_3 &= \left(\frac{u_3+v_3}{2}\right)t_3 \\ \Rightarrow s &= 48t. \\ s_4 &= \left(\frac{u_4+v_4}{2}\right)t_4 \\ \Rightarrow 6-s &= \left(\frac{48+0}{2}\right)\left(\frac{1}{6}-t\right) \\ &= 4-24t \\ \Rightarrow 2+24t &= s. \end{aligned}$$

Letting the two expressions for s equal each other,

$$\begin{aligned} 48t &= 2+24t \\ \Rightarrow 24t &= 2 \\ \Rightarrow t &= \frac{1}{12} \text{ hours} \\ \Rightarrow s &= 4 \text{ km}. \end{aligned}$$

Our UVAST array is now

First Part	Second Part	Third Part	Fourth Part	Extra Equations
$u_1 = 0$	$u_2 = 32$	$u_3 = 48$	$u_4 = 48$	$\cancel{t_3 + t_4 = \frac{1}{6}}$
$v_1 = 32$	$v_2 = 48$	$v_3 = 48$	$v_4 = 0$	$\cancel{s_1 + s_2 + s_3 + s_4 = 20}$
$a_1 =$	$a_2 =$	$a_3 = 0$	$a_4 =$	
$s_1 = 4$	$s_2 = 10$	$s_3 = \cancel{s}\ 4$	$s_4 = \cancel{6-s}\ 2$	
$t_1 = \frac{1}{4}$	$t_2 = \frac{1}{4}$	$t_3 = \cancel{t}\ \frac{1}{12}$	$t_4 = \cancel{\frac{1}{6}-t}\ \frac{1}{12}$	

Therefore we reach the 16 km mark during the third part of the journey. Specifically halfway through it, as we are at 14 km when we start this part of the journey and at 18 km at the end. Therefore the time taken for the first 16 km is

$$\frac{1}{4}+\frac{1}{4}+\frac{1}{2}\left(\frac{1}{12}\right) = \frac{13}{24} \text{ hours.}$$

Note that we can write our final answers in minutes (so in this case 32.5 minutes), we just can't mix units when doing our calculations. The question asks for the time taken, not the amount of minutes/hours so any units are fine.

(iii) The deceleration in the fourth part of the journey can easily be found using any equation of motion.

$$\begin{aligned} v &= u+at \\ 0 &= 48+\frac{a}{12} \\ \Rightarrow a &= -576 \text{ km/hr}^2. \end{aligned}$$

So the train decelerates at a rate of 576 km/hr^2 in the final part of the journey.

Example 1.84 — 2022 Q1 (b).

(b) A ball E is thrown vertically upwards with a speed of 42 m s^{-1}.

T (< 8) seconds later another ball, F, is thrown vertically upwards from the same point with the same initial speed.

(i) Find where ball E is after 5 s and the total distance it has travelled in this time.

(ii) Prove that when E and F collide, they will each be travelling with speed $\frac{1}{2}gT$.

Figure 1.15

(i) This question doesn't require ball F, so if we're looking at the first 5 seconds of ball E's journey our UVAST array just looks like this.

$$
\begin{aligned}
u &= 42 \\
v &= \\
a &= -g \\
s &= \\
t &= 5.
\end{aligned}
$$

To get the position of the ball,

$$
\begin{aligned}
s &= ut + \frac{1}{2}at^2 \\
\Rightarrow s &= 42(5) - \frac{g}{2}(5)^2 \\
\Rightarrow s &= 87.5 \text{ m.}
\end{aligned}
$$

So ball E is 87.5 metres above where it was thrown. To find the total distance it has travelled, we need to know if it has changed direction, and if so what its greatest height was. See that

$$
\begin{aligned}
v &= u + at \\
\Rightarrow v &= 42 - 5g \\
&= -7.
\end{aligned}
$$

As final velocity is negative the ball has changed direction by time $t = 5$. Therefore we need to find the greatest height, which we can do using a different UVAST array:

$$
\begin{aligned}
u &= 42 \\
v &= 0 \\
a &= -g \\
s &= \\
t &=
\end{aligned}
$$

Then

$$\begin{aligned} v^2 &= u^2 + 2as \\ \Rightarrow 0 &= 42^2 - 2gs \\ \Rightarrow s &= \frac{42^2}{2g} \\ &= 90 \text{ m}. \end{aligned}$$

So the ball travelled up 90 m, then downwards $90 - 87.5 = 2.5$ m so that it's total distance travelled is 92.5 m.

(ii) The UVAST array below is for the journeys of balls E and F that start when they are thrown and end when they collide.

Ball E	Ball F	Extra Equations
$u_E = 42$	$u_F = 42$	$t_E = t_F + T$
$v_E =$	$v_F =$	$s_E = s_F$
$a_E = -g$	$a_F = -g$	
$s_E =$	$s_F =$	
$t_E =$	$t_F =$	

Letting $t_F = t$, $s_F = s$, our UVAST array can then be written as

Ball E	Ball F	Extra Equations
$u_E = 42$	$u_F = 42$	~~$t_E = t_F + T$~~
$v_E =$	$v_F =$	~~$s_E = s_F$~~
$a_E = -g$	$a_F = -g$	
$s_E = s$	$s_F = s$	
$t_E = t + T$	$t_F = t$	

Because the balls were thrown at the same speed, when they collide Ball E is travelling downwards and Ball F is travelling upwards. Therefore we want to show that

$$\begin{aligned} v_E &= -\frac{1}{2}gT, \\ v_F &= \frac{1}{2}gT. \end{aligned}$$

As per Rule 1.68, we can treat T as an equation variable and s, t as non-equation variables. From the first column,

$$\begin{aligned} s_E &= u_E t_E + \frac{1}{2}a_E t_E^2 \\ \Rightarrow s &= 42(t+T) - \frac{g}{2}(t+T)^2 \end{aligned}$$

so that we have s in terms of other variables. From the second column,

$$
\begin{aligned}
s_F &= u_F t_F + \frac{1}{2} a_F t_F^2 \\
\Rightarrow 42(t+T) - \frac{g}{2}(t+T)^2 &= 42t - \frac{g}{2}t^2 \\
\Rightarrow 42t + 42T - \frac{g}{2}(t^2 + 2tT + T^2) &= 42t - \frac{g}{2}t^2 \\
\Rightarrow 42t + 42T - \frac{g}{2}t^2 - gtT - \frac{g}{2}T^2 &= 42t - \frac{g}{2}t^2 \\
\Rightarrow 42T - gtT - \frac{g}{2}T^2 &= 0 \\
\Rightarrow 42T - \frac{g}{2}T^2 &= gtT \\
\Rightarrow \frac{42}{g} - \frac{T}{2} &= t.
\end{aligned}
$$

Now that we have t in terms of only the equation variable T we can do similarly for s.

$$
\begin{aligned}
s &= 42(t+T) - \frac{g}{2}(t+T)^2 \\
&= 42\left(\frac{42}{g} - \frac{T}{2} + T\right) - \frac{g}{2}\left(\frac{42}{g} - \frac{T}{2} + T\right)^2 \\
&= 42\left(\frac{42}{g} - \frac{T}{2}\right) - \frac{g}{2}\left(\frac{42}{g} - \frac{T}{2}\right)^2 \\
&= \frac{1764}{g} - 21T - \frac{g}{2}\left(\frac{1764}{g^2} - \frac{42}{g}T + \frac{T^2}{4}\right) \\
&= \frac{1764}{g} - 21T - \frac{882}{g} + 21T - \frac{gT^2}{8} \\
&= \frac{882}{g} - \frac{gT^2}{8}.
\end{aligned}
$$

Our UVAST array now looks as follows.

Ball E	Ball F	Extra Equations
$u_E = 42$	$u_F = 42$	~~$t_E = t_F + T$~~
$v_E =$	$v_F =$	~~$s_E = s_F$~~
$a_E = -g$	$a_F = -g$	
$s_E = \cancel{s}\ \frac{882}{g} - \frac{gT^2}{8}$	$s_F = \cancel{s}\ \frac{882}{g} - \frac{gT^2}{8}$	
$t_E = \cancel{t+T}\ \frac{42}{g} + \frac{T}{2}$	$t_F = \cancel{t}\ \frac{42}{g} - \frac{T}{2}$	

We can now find v_E, v_F by applying any equation of motion which contains v to each column.

$$\begin{aligned} v_E &= u_E + a_E t_E \\ \Rightarrow v_E &= 42 - g\left(\frac{42}{g} + \frac{T}{2}\right) \\ &= -\frac{gT}{2}. \end{aligned}$$

Therefore the speed of ball E is

$$\left|-\frac{gT}{2}\right| = \frac{gT}{2},$$

as required. Similarly

$$\begin{aligned} v_F &= u_F + a_F t_F \\ \Rightarrow v_F &= 42 - g\left(\frac{42}{g} + \frac{T}{2}\right) \\ &= \frac{gT}{2}. \end{aligned}$$

Question 1.85 — 2016 Q1.

(a) A car has an initial speed of u m s^{-1}. It moves in a straight line with constant acceleration f for 4 seconds. It travels 40 m while accelerating.
The car then moves with uniform speed and travels 45 m in 3 seconds.
It is then brought to rest by a constant retardation $2f$.

(i) Draw a speed-time graph for the motion.

(ii) Find the value of u.

(iii) Find the total distance travelled.

(b) A particle is projected vertically upwards with a velocity of u m s^{-1}.
After an interval of $2t$ seconds a second particle is projected vertically upwards from the same point and with the same initial velocity.

They meet at a height of h m.

Show that $h = \dfrac{u^2 - g^2t^2}{2g}$.

Figure 1.16

Note 1.86 The methods given here are not the only way to tackle Linear Motion problems. Other methods exist and some students may find themselves drawn to them. In particular, writing down UVAST arrays as we have done is something that every student should do at the beginning of a question. There is no alternative to this. However, Rules 1.37 and 1.68 are unique to these notes. They are designed as general sets of steps (a Grand Unified Theory if you will) that any student can apply to almost any problem. They are quite formal, and sometimes not the quickest way to solve problems (for example, in Exercise 1.84 we didn't need to find s in terms of T, this part could have been skipped). Some students may prefer more intuitive approaches, where they simply play around with the variables until they get what they want. Some students may mostly stick to some version of these rules, but find their own style as they practice. Part of the purpose of practicing problems like this is not just to "learn the method". but also find out what works for you.

1.11 Summary

- Most problems asking for numerical answers, whether they involve one or multiple objects, involve setting up a UVAST array with extra equations, replacing those equations with variables, solving for those variables by setting up simultaneous equations or by eliminating them from the UVAST array one by one, and then answering the question asked, as outlined in Rules 1.37 and 1.58.

- For problems asking you to prove an equation, the approach is similar. Set up a UVAST array with extra equations (which will contain variables and numbers), replace those equations with variables, get the non-equation variables in terms of the equation variables using simultaneous equations or by finding them in terms of other variables one by one, set up an equation that only contains equation variables and rearrange it to the form required in the question, as outlined in Rule 1.68.

- There is never a need to use the same equation twice on a column of a UVAST array, nor to use more than two equations on one column (see Rules 1.37 and 1.68).

- When you are asked for the value of a variable and find two values, you should find a reason that one of the values is invalid. This may involve re-reading the question.

- Be aware of units used in questions; not all quantities are given in the same units. See Homework Questions 9 and 11, for example.

- Give units in all of your answers.

- In freefall problems, acceleration due to gravity is $g = 9.8$ m/s^2. It is usually advised to leave g as a variable (like π) rather than replacing it with its decimal value, unless the question explicitly asks you to.

- Remember the basics of freefall problems as outlined in Note 1.20.

- When setting up a problem, particularly those involving more than one object or freefall problems, decide which is the positive direction. Also decide what the entire journey is (see Question 1.27).

- Remember that while velocity and displacement can be negative, speed and distance cannot be. Be careful how questions are phrased, i.e. whether you are asked for the velocity or the speed of an object for example.

- For problems involving more than one object, there is almost always a connection between the time and displacement quantities of the journey of each object.

- In more algebraic problems, remember that when you are told that the **deceleration** of an object is d, its **acceleration** is $-d$, and this is the expression that goes in your UVAST array.

- Speed is the absolute value of velocity. If an object never changes direction, its distance travelled is the absolute value of its displacement.

- In time-velocity graphs, only the major quantities need to be marked. That is, all time intervals, and all heights.
- In time-velocity graphs, the area under the graph is equal to the total distance travelled.
- In time-displacement graphs, be able to study the three properties of the graph (related to displacement, velocity and acceleration) and use them to build up a story of the journey the graph describes.

1.12 Notes on the Exam, and Work Still to Cover

On the old syllabus exam (up to 2022), Linear Motion was Question 1 of the paper.

Linear motion has not changed much under the new syllabus, both on the syllabus in theory and in exam questions in practice. The only real change in theory is that displacement-time graphs are new to the new syllabus (although they did not come up on the sample paper or 2023 paper), and in practice students were given axes when asked to draw velocity-time graphs.

In the last 10 years, many Linear Motion questions on the Leaving Cert exam have required knowledge from other topics. Specifically they have required knowledge of Connected Particles, Collisions, Work Energy & Power, and Circular Motion. As a result students revising exam questions may find some requiring knowledge they don't have, and so in the Revision section of this chapter a list of exam questions that students should not yet attempt are listed. This means that we will revisit Linear Motion exam questions later in the year.

1.13 Homework

Five Quantities, Five Equations

1. A car notices a speed camera 150 m ahead. The driver decelerates uniformly at a rate of 3 m/s until passes the speed camera at a speed of 40 m/s. What was the driver's initial speed?
2. A cyclist accelerates at 4 m/s from an initial speed of 10 m/s. How long does it take to cover 100 m?
3. A car accelerates uniformly from 2 m/s to 12 m/s over 4 seconds. Calculate the distance travelled in this time.
4. A racecar, starting from rest, accelerates uniformly at a rate of 8 m/s^2. It passes a checkpoint 100 m from its starting position. At what speed does it pass this checkpoint?

Freefall

5. An object at ground level is thrown vertically in the air at a speed of 25 m/s. Giving your answers in terms of g,
   (a) What height will it reach?
   (b) How much time will pass between when the ball is thrown and when it hits the ground?
   (c) What is the total distance that the object has travelled by the time it hits the ground?
6. An object is dropped from the top of a building of height 20 m. Giving your answers in terms of g,
   (a) How long will it take the object to hit the ground?
   (b) At what speed will it hit the ground?
7. An object is thrown vertically upwards with a speed of 20 m/s. Giving your answers to two decimal places,
   (a) In what direction is it travelling after 3 seconds?
   (b) How high is it off the ground after 3 seconds?
   (c) How much distance has it travelled in 3 seconds?

Multi-Part Journeys

8. A sprint runner is in a 100 m race. He can accelerate at a rate of 15 m/s^2 to a maximum speed of 10 m/s. How long should it take him to finish the race if he starts from rest?
9. A car starts from rest with an acceleration of 4 m/s^2. After covering 100 m it travels at a constant velocity for 2 minutes, then slows to rest with a deceleration of 6 m/s^2. What is its average speed for the whole journey?
10. A racecar is travelling at its maximum speed of 80 m/s, and the driver notices a turn 500 m ahead. The driver estimates that she needs to enter the turn at a speed of 20 m/s. The racecar can decelerate at a rate of 10 m/s^2. After how many seconds does the driver need to start

decelerating? In that case, how long does it take to reach the turn?

More Complex Multi-Part Journeys

11. A train has to get between two stations 1 km apart, and it must stop at both stations. It can accelerate at a rate of 2 m/s^2 and decelerate at a rate of 3 m/s^2. How quickly can the train get from one station to the other if
    (a) it is subject to a speed limit of 20 m/s,
    (b) there is no speed limit?

12. A motorbike moves onto the motorway at a speed of 20 m/s, and notices a speed camera 250 m ahead. The speed limit is 40 m/s and the bike can both accelerate and decelerate at a rate of 5 m/s^2. How quickly can the motorbike get to the speed camera by
    (a) Never going over the speed limit
    (b) Not being over the speed limit at the point it passes the camera? Ignore the notion of a maximum speed

Velocity-Time Graphs

13. For Homework Questions 1-4, 8-12, draw a time-velocity graph of the motion. Only use quantitites given in the question, not ones you calculated yourself. Your graph should therefore have many blank quantities; what is important is drawing the correct shape.

Multi-Object Journeys

14. A car starts from rest with an acceleration of 4 m/s^2, and at the same time a bus passes the car, travelling with a speed of 10 m/s and acceleration 2 m/s^2.
    (a) After how many seconds does the car pass out the truck?
    (b) What is the difference in velocity between the car and the truck when the car passes out the truck?

15. Two racecars are at the same point, travelling towards the finish line of a racing track. Racecar A has acceleration 5 m/s^2 and velocity 50 m/s, and racecar B has acceleration 3 m/s^2 and velocity 60 m/s. How close does the finish line have to be for racecar B to finish first?

16. Two trains 3 km apart are travelling toward each other. Train A has a velocity of 50 km/hr and acceleration 3 km/hr^2, and train B has a velocity of 30 km/hr and an acceleration of 1.5 km/hr^2. After how many seconds do they meet?

17. Two racecars leave a corner at different speeds and at different times. Racecar *A* leaves the corner 5 seconds before Racecar *B*, with a speed of 50 m/s and an acceleration of 8 m/s^2, while Racecar *B* leaves with a speed of 40 m/s and an acceleration of 12 m/s^2. How long after Racecar *B* leaves the corner will it catch up with Racecar *A*?

More Algebraic Problems

18. A car travels past a point P with velocity u. By the time it reaches the point Q, its speed has increased to $2u$, and by the time it reaches point R, its speed is $3u$. It has the same acceleration for the entire journey. What is $|PQ| : |QR|$ (the ratio of the distance between P and Q and the distance between Q and R)?

19. A car starts from rest with an acceleration a to its maximum speed, it then immediately decelerates with rate $2a$ to rest. The total time of the journey is 18 seconds. Find the total distance covered and the maximum speed in terms of a.

20. Two points P and Q are a distance d apart. A particle starts at P and moves towards Q with initial velocity $2u$ and uniform acceleration a. A second particle starts at the same time from Q and moves towards P with initial velocity $3u$ and uniform **deceleration** a. Prove that the particles collide after $\frac{d}{5u}$ seconds.

1.14 Homework Solutions

Five Quantities, Five Equations

1.

$$\begin{aligned} u &= \\ v &= 40 \\ a &= -3 \\ s &= 150 \\ t &= \end{aligned}$$

$$\begin{aligned} v^2 &= u^2 + 2as \\ \Rightarrow 40^2 &= u^2 + 2(-3)(150) \\ \Rightarrow 1600 &= u^2 - 900 \\ \Rightarrow 2500 &= u^2 \\ \Rightarrow 50 \text{ m/s} &= u. \end{aligned}$$

2.

$$\begin{aligned} u &= 10 \\ v &= \\ a &= 4 \\ s &= 100 \\ t &= \end{aligned}$$

$$\begin{aligned} s &= ut + \frac{1}{2}at^2 \\ \Rightarrow 100 &= 10t + 2t^2 \\ \Rightarrow 0 &= t^2 + 5t - 50 \\ \Rightarrow 0 &= (t-5)(t+10) \\ \Rightarrow t &= \cancel{-10}, 5 \text{ seconds}. \end{aligned}$$

3.

$$\begin{aligned} u &= 2 \\ v &= 12 \\ a &= \\ s &= \\ t &= 4 \end{aligned}$$

$$\begin{aligned} s &= \left(\frac{u+v}{2}\right)t \\ \Rightarrow s &= \left(\frac{2+12}{2}\right)4 \\ &= 28 \text{ m.} \end{aligned}$$

4.

$$\begin{aligned} u &= 0 \\ v &= \\ a &= 8 \\ s &= 100 \\ t &= \end{aligned}$$

$$\begin{aligned} v^2 &= u^2 + 2as \\ &= 2(8)(100) \\ &= 1600 \\ \Rightarrow v &= 40 \text{ m/s.} \end{aligned}$$

Freefall

5. (a) With upwards direction positive,

$$\begin{aligned} u &= 25 \\ v &= 0 \\ a &= -g \\ s &= \\ t &= \end{aligned}$$

$$\begin{aligned} v^2 &= u^2 + 2as \\ \Rightarrow 0^2 &= 25^2 - 2gs \\ \Rightarrow 2gs &= 625 \\ \Rightarrow s &= \frac{625}{2g} \\ &\approx 31.89 \text{ m.} \end{aligned}$$

(b)

$$\begin{aligned} u &= 25 \\ v &= \\ a &= -g \\ s &= 0 \\ t &= \end{aligned}$$

$$
\begin{aligned}
s &= ut + \frac{1}{2}at^2 \\
\Rightarrow 0 &= 25t - \frac{g}{2}t^2 \\
\Rightarrow 0 &= 25 - \frac{g}{2}t \quad \text{(as } t \neq 0\text{)} \\
\Rightarrow \frac{g}{2}t &= 25 \\
\Rightarrow t &= \frac{50}{g} \text{ seconds} \\
&\approx 5.1 \text{ seconds.}
\end{aligned}
$$

(c)

$$
\begin{aligned}
\text{Total distance travelled} &= 2\left(\frac{625}{2g}\right) \\
&= 63.78 \text{ m.}
\end{aligned}
$$

6. (a) With downwards direction positive,

$$
\begin{aligned}
u &= 0 \\
v &= \\
a &= g \\
s &= 20 \\
t &=
\end{aligned}
$$

$$
\begin{aligned}
s &= ut + \frac{1}{2}at^2 \\
\Rightarrow 20 &= \frac{g}{2}t^2 \\
\Rightarrow \frac{40}{g} &= t^2 \\
\Rightarrow \sqrt{\frac{40}{g}} \text{ seconds} &= t.
\end{aligned}
$$

(b)

$$
\begin{aligned}
v &= u + at \\
\Rightarrow v &= g\sqrt{\frac{40}{g}} \\
&= \sqrt{g^2}\sqrt{\frac{40}{g}} \\
&= \sqrt{g^2\frac{40}{g}} \\
&= \sqrt{40g} \text{ m/s.}
\end{aligned}
$$

7. (a) With upwards direction positive,

$$
\begin{aligned}
u &= 20 \\
v &= \\
a &= -g \\
s &= \\
t &= 3
\end{aligned}
$$

$$
\begin{aligned}
v &= u + at \\
&= 20 - 3g \\
&= -9.4 \text{ m/s} \\
&< 0
\end{aligned}
$$

so after 3 seconds it's travelling downwards, or towards the ground.

(b)

$$
\begin{aligned}
s &= ut + \frac{1}{2}at^2 \\
&= 20(3) - \frac{g}{2}(3)^2 \\
&= 15.9 \text{ m.}
\end{aligned}
$$

(c) We have to find its maximum height.

$$
\begin{aligned}
u &= 20 \\
v &= 0 \\
a &= -g \\
s &= \\
t &=
\end{aligned}
$$

$$
\begin{aligned}
v^2 &= u^2 + 2as \\
\Rightarrow 0^2 &= 20^2 - 2gs \\
\Rightarrow 2gs &= 400 \\
\Rightarrow s &= \frac{200}{g} \\
\Rightarrow \text{Answer} &= \frac{200}{g} + \left(\frac{200}{g} - 15.9\right) \\
&\approx 24.92 \text{ m.}
\end{aligned}
$$

Multi-Part Journeys

8.

First Part	Second Part	Extra Equations
$u_1 = 0$	$u_2 = 10$	$s_1 + s_2 = 100$
$v_1 = 10$	$v_2 = 10$	
$a_1 = 15$	$a_2 = 0$	
$s_1 =$	$s_2 =$	
$t_1 =$	$t_2 =$	

$$
\begin{aligned}
v_1^2 &= u_1^2 + 2a_1 s_1 \\
\Rightarrow 10^2 &= 0^2 + 30s_1 \\
\Rightarrow 100 &= 30s_1 \\
\Rightarrow \frac{10}{3} &= s_1 \\
\Rightarrow s_2 &= 100 - \frac{10}{3} \\
&= \frac{290}{3}. \\
v_1 &= u_1 + a_1 t_1 \\
\Rightarrow 10 &= 0 + 15t_1 \\
\Rightarrow \frac{2}{3} \text{ seconds} &= t_1. \\
s_2 &= \left(\frac{u_2 + v_2}{2}\right) t_2 \\
\Rightarrow \frac{290}{3} &= 10t_2 \\
\Rightarrow \frac{29}{3} \text{ seconds} &= t_2. \\
\text{Total Time} &= \frac{10}{3} + \frac{29}{3} \\
&= \frac{31}{3} \text{ seconds.}
\end{aligned}
$$

9.

First Part	Second Part	Third Part	Extra Equations
$u_1 = 0$	$u_2 =$	$u_3 =$	$v_1 = u_2$
$v_1 =$	$v_2 =$	$v_3 = 0$	$v_2 = u_3$
$a_1 = 4$	$a_2 = 0$	$a_3 = -6$	
$s_1 = 100$	$s_2 =$	$s_3 =$	
$t_1 =$	$t_2 = 120$	$t_3 =$	

$$
\begin{aligned}
v_1^2 &= u_1^2 + 2a_1 s_1 \\
\Rightarrow v_1^2 &= 0^2 + 2(4)100 \\
\Rightarrow v_1^2 &= 800 \\
\Rightarrow v_1 &= \sqrt{800}.
\end{aligned}
$$

As $v_1 = \sqrt{800}$, $u_2, v_2, u_3 = \sqrt{800}$ also, and basic calculations give s_2, s_3, t_1, t_3. Then

$$\begin{aligned}\text{Average Speed} &= \frac{s_1+s_2+s_3}{t_1+t_2+t_3}\\ &= \frac{100+120\sqrt{800}+\frac{200}{3}}{\frac{\sqrt{800}}{4}+120+\frac{\sqrt{800}}{6}}\\ &\approx 27.02 \text{ m/s.}\end{aligned}$$

10.

First Part	Second Part	Extra Equations
$u_1 = 80$	$u_2 = 80$	$s_1 + s_2 = 500$
$v_1 = 80$	$v_2 = 20$	
$a_1 = 0$	$a_2 = -10$	
$s_1 =$	$s_2 =$	
$t_1 =$	$t_2 =$	

$$\begin{aligned}v_2 &= u_2 + a_2 t_2\\ \Rightarrow 20 &= 80 - 10t_2\\ \Rightarrow -60 &= -6t_2\\ \Rightarrow t_2 &= 6.\\ s_2 &= \left(\frac{u_2+v_2}{2}\right)t_2\\ &= \left(\frac{80+20}{2}\right)6\\ &= 300.\\ s_1 &= 500 - s_2\\ &= 200.\\ s_1 &= \left(\frac{u_1+v_1}{2}\right)t_1\\ \Rightarrow 200 &= 80t_1\\ \Rightarrow \frac{5}{2}\text{ seconds} &= t_1\\ \Rightarrow \text{Total time} &= t_1 + t_2\\ &= \frac{17}{2}\text{ seconds.}\end{aligned}$$

More Complex Multi-Part Journeys

11. (a)

First Part	Second Part	Third Part	Extra Equations
$u_1 = 0$	$u_2 = 20$	$u_3 = 20$	$s_1 + s_2 + s_3 = 1000$
$v_1 = 20$	$v_2 = 20$	$v_3 = 0$	
$a_1 = 2$	$a_2 = 0$	$a_3 = -3$	
$s_1 =$	$s_2 =$	$s_3 =$	
$t_1 =$	$t_2 =$	$t_3 =$	

$$\begin{aligned} v_1^2 &= u_1^2 + 2a_1 s_1 \\ \Rightarrow 20^2 &= 2(2)s_1 \\ \Rightarrow 400 &= 4s_1 \\ \Rightarrow 100 &= s_1. \end{aligned}$$

$$\begin{aligned} v_3^2 &= u_3^2 + 2a_3 s_3 \\ \Rightarrow 0^2 &= 20^2 + 2(-3)s_1 \\ \Rightarrow 0 &= 400 - 6s_1 \\ \Rightarrow 6s_1 &= 400 \\ \Rightarrow s_1 &= \frac{200}{3} \\ \Rightarrow s_2 &= 1000 - s_1 - s_3 \\ &= \frac{2500}{3}. \end{aligned}$$

Basic calculations from here give t_1, t_2, t_3, so that

$$\begin{aligned} \text{Total Time} &= t_1 + t_2 + t_3 \\ &= 10 + \frac{125}{3} + \frac{20}{3} \\ &= \frac{175}{3} \text{ seconds.} \end{aligned}$$

(b)

First Part	Second Part	Extra Equations
$u_1 = 0$	$u_2 = v$	~~$s_1 + s_2 = 1000$~~
$v_1 = v$	$v_2 = 0$	~~$v_1 = u_2$~~
$a_1 = 2$	$a_2 = -3$	
$s_1 = s$	$s_2 = 1000 - s$	
$t_1 =$	$t_2 =$	

$$\begin{aligned} v_1^2 &= u_1^2 + 2a_1 s_1 \\ \Rightarrow v^2 &= 4s. \\ v_2^2 &= u_2^2 + 2a_2 s_2 \\ \Rightarrow 0 &= v^2 - 6(1000 - s) \\ \Rightarrow 0 &= v^2 - 6000 + 6s \\ 6000 - 6s &= v^2. \end{aligned}$$

Setting the v^2 expressions equal to each other,

$$
\begin{aligned}
4s &= 6000 - 6s \\
\Rightarrow 10s &= 6000 \\
\Rightarrow s &= 600 \\
\Rightarrow v^2 &= 4(600) \\
&= 2400 \\
\Rightarrow v &= \sqrt{2400}. \\
v_1 &= u_1 + a_1 t_1 \\
\Rightarrow \sqrt{2400} &= 2t_1 \\
\Rightarrow \frac{\sqrt{2400}}{2} &= t_1. \\
v_2 &= u_2 + a_2 t_2 \\
\Rightarrow 0 &= \sqrt{2400} - 3t_2 \\
\Rightarrow 3t_2 &= \sqrt{2400} \\
t_2 &= \frac{\sqrt{2400}}{3} \\
\Rightarrow \text{Total time} &= t_1 + t_2 \\
&= \frac{5\sqrt{2400}}{6} \text{ seconds} \\
&= \frac{100}{\sqrt{6}} \text{ seconds} \\
&\approx 40.82 \text{ seconds.}
\end{aligned}
$$

12. (a)

First Part	Second Part	Extra Equations
$u_1 = 20$	$u_2 = 40$	$s_1 + s_2 = 250$
$v_1 = 40$	$v_2 = 40$	
$a_1 = 5$	$a_2 = 0$	
$s_1 =$	$s_2 =$	
$t_1 =$	$t_2 =$	

$$\begin{aligned}
v_1^2 &= u_1^2 + 2a_1 s_1 \\
\Rightarrow 1600 &= 400 + 10s_1 \\
\Rightarrow 120 &= s_1 \\
\Rightarrow s_2 &= 250 - s_1 \\
&= 130. \\
v_1 &= u_1 + a_1 t_1 \\
\Rightarrow 40 &= 20 + 5t_1 \\
\Rightarrow 4 &= t_1. \\
s_2 &= \left(\frac{u_2 + v_2}{2}\right) t_2 \\
\Rightarrow 130 &= 40t_2 \\
\Rightarrow \frac{13}{4} &= t_2 \\
\Rightarrow \text{Total time} &= t_1 + t_2 \\
&= \frac{29}{4} \text{ seconds.}
\end{aligned}$$

(b)

First Part	Second Part	Extra Equations
$u_1 = 20$	$u_2 = v$	~~$s_1 + s_2 = 250$~~
$v_1 = v$	$v_2 = 40$	~~$v_1 = u_2$~~
$a_1 = 5$	$a_2 = -5$	
$s_1 = s$	$s_2 = 250 - s$	
$t_1 =$	$t_2 =$	

$$\begin{aligned}
v_1^2 &= u_1^2 + 2a_1 s_1 \\
\Rightarrow v^2 &= 400 + 10s. \\
v_2^2 &= u_2^2 + 2a_2 s_2 \\
\Rightarrow 1600 &= v^2 - 10(250 - s) \\
\Rightarrow 1600 &= v^2 - 2500 + 10s \\
\Rightarrow 4100 - 10s &= v^2.
\end{aligned}$$

Letting the expressions for v^2 equal each other,

$$\begin{aligned}
400 + 10s &= 4100 - 10s \\
20s &= 3700 \\
\Rightarrow s &= 185 \\
\Rightarrow v^2 &= 400 + 10(185) \\
&= 2250 \\
\Rightarrow v &= \sqrt{2250}.
\end{aligned}$$

From here basic calculations give $t_1 = \frac{\sqrt{2250}-20}{5}$, $t_2 = \frac{\sqrt{2250}-40}{5}$ so that

$$\begin{aligned}
\text{Total time} &= \frac{\sqrt{2250} - 20}{5} + \frac{\sqrt{2250} - 40}{5} \\
&= \frac{2\sqrt{2250} - 60}{5} \text{ seconds.}
\end{aligned}$$

Velocity-Time Graphs

13.

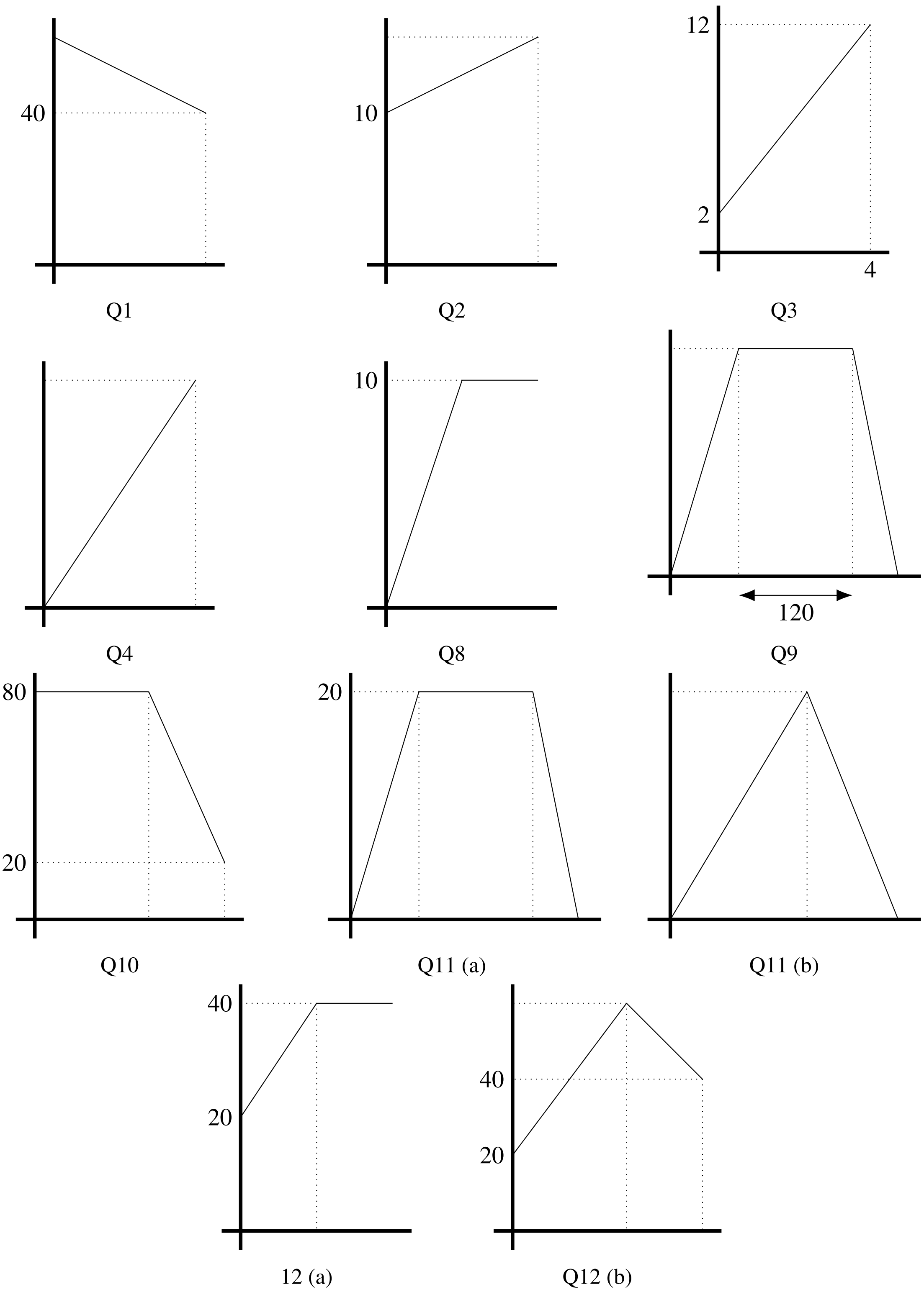


Figure 1.17

Multi-Object Journeys

14. (a)

Car	Bus	Extra Equations
$u_C = 0$	$u_B = 10$	~~$s_C = s_B$~~
$v_C =$	$v_B =$	~~$t_C = t_B$~~
$a_C = 4$	$a_B = 2$	
$s_C = s$	$s_B = s$	
$t_C = t$	$t_B = t$	

$$\begin{aligned} s_C &= u_C t_C + \frac{1}{2} a_C t_C^2 \\ \Rightarrow s &= 2t^2. \\ s_B &= u_B t_B + \frac{1}{2} a_B t_B^2 \\ \Rightarrow s &= 10t + t^2. \end{aligned}$$

Setting the expressions for s equal to each other,

$$\begin{aligned} 2t^2 &= 10t + t^2 \\ \Rightarrow t^2 &= 10t \\ \Rightarrow t &= 10 \text{ seconds.} \end{aligned}$$

(b) Basic calculations give $v_C = 40$, $v_C = 30$ so that

$$\begin{aligned} \text{Difference in speed} &= v_C - v_B \\ &= 40 - 30 \\ &= 10 \text{ m/s.} \end{aligned}$$

15. Assuming line is close enough that A just catches up at finish:

Car A	Car B	Extra Equations
$u_A = 50$	$u_B = 60$	~~$s_A = s_B$~~
$v_A =$	$v_B =$	~~$t_A = t_B$~~
$a_A = 5$	$a_B = 3$	
$s_A = s$	$s_B = s$	
$t_A = t$	$t_B = t$	

$$\begin{aligned} s_1 &= u_1 t_1 + \frac{1}{2} a_1 t_1^2 \\ \Rightarrow s &= 50t + \frac{5}{2} t^2. \\ s_2 &= u_2 t_2 + \frac{1}{2} a_2 t_2^2 \\ \Rightarrow s &= 60t + \frac{3}{2} t^2. \end{aligned}$$

Setting the s expressions equal,

$$\begin{aligned} 50t + \frac{5}{2}t^2 &= 60t + \frac{3}{2}t^2 \\ \Rightarrow t^2 &= 10t \\ \Rightarrow t &= 10 \text{ seconds} \\ \Rightarrow s &= 50(10) + \frac{5}{2}(10)^2 \\ &= 750 \text{ m.} \end{aligned}$$

So the finish line has to be < 750 m for racecar B to finish first.

16.

Train A	Train B	Extra Equations
$u_A = 50$	$u_B = 30$	~~$s_A + s_B = 3$~~
$v_A =$	$v_B =$	~~$t_A = t_B$~~
$a_A = 3$	$a_B = \frac{3}{2}$	
$s_A = s$	$s_B = 3 - s$	
$t_A = t$	$t_B = t$	

$$\begin{aligned} s_A &= u_A t_A + \frac{1}{2}a_A t_A^2 \\ \Rightarrow s &= 50t + \frac{3}{2}t^2. \\ s_B &= u_B t_B + \frac{1}{2}a_B t_B^2 \\ \Rightarrow 3 - s &= 30t + \frac{3}{4}t^2 \\ \Rightarrow -s &= -3 + 30t + \frac{3}{4}t^2 \\ \Rightarrow s &= 3 - 30t - \frac{3}{4}t^2. \end{aligned}$$

Setting the s expressions equal,

$$\begin{aligned} 3 - 30t - \frac{3}{4}t^2 &= 50t + \frac{3}{2}t^2 \\ \Rightarrow 0 &= \frac{9}{4}t^2 + 80t - 3 \\ \Rightarrow t &= \cancel{-35.593},\ 0.0374605 \text{ hours} \\ &= 134.86 \text{ seconds.} \end{aligned}$$

17.

Car A	Car B	Extra Equations
$u_A = 50$	$u_B = 40$	~~$s_A = s_B$~~
$v_A =$	$v_B =$	~~$t_A = t_B + 5$~~
$a_A = 8$	$a_B = 12$	
$s_A = s$	$s_B = s$	
$t_A = t + 5$	$t_B = t$	

$$
\begin{aligned}
s_A &= u_A t_A + \frac{1}{2} a_A t_A^2 \\
\Rightarrow s &= 50(t+5) + 4(t+5)^2 \\
&= 50t + 250 + 4(t^2 + 10t + 25) \\
&= 4t^2 + 90t + 350. \\
s_B &= u_B t_B + \frac{1}{2} a_B t_B^2 \\
\Rightarrow s &= 40t + 6t^2.
\end{aligned}
$$

Setting the s expressions equal,

$$
\begin{aligned}
40t + 6t^2 &= 4t^2 + 90t + 350 \\
\Rightarrow 2t^2 - 50t - 350 &= 0 \\
\Rightarrow t &= \cancel{-5.7}, 30.7 \text{ seconds.}
\end{aligned}
$$

More Algebraic Problems

18. First part of journey is PQ, second part is QR.

First Part	Second Part	Extra Equations
$u_1 = u$	$u_2 = 2u$	~~$a_1 = a_2$~~
$v_1 = 2u$	$v_2 = 3u$	
$a_1 = a$	$a_2 = a$	
$s_1 =$	$s_2 =$	
$t_1 =$	$t_2 =$	

$$
\begin{aligned}
v_1^2 &= u_1^2 + 2a_1 s_1 \\
\Rightarrow 4u^2 &= u^2 + 2as_1 \\
\Rightarrow \frac{3u^2}{2a} &= s_1. \\
v_2^2 &= u_2^2 + 2a_2 s_2 \\
\Rightarrow 9u^2 &= 4u^2 + 2as_2 \\
\frac{5u^2}{2a} &= s_1 \\
\Rightarrow |PQ| : |QR| &= \frac{3u^2}{2a} : \frac{5u^2}{2a} \\
&= 3 : 5.
\end{aligned}
$$

19.

First Part	Second Part	Extra Equations
$u_1 = 0$	$u_2 = v$	~~$u_1 = v_1$~~
$v_1 = v$	$v_2 = 0$	~~$t_1 + t_2 = 18$~~
$a_1 = a$	$a_2 = -2a$	
$s_1 =$	$s_2 =$	
$t_1 = t$	$t_2 = 18 - t$	

$$\begin{aligned}
v_1 &= u_1 + a_1 t_1 \\
\Rightarrow v &= at. \\
v_2 &= u_2 + a_2 t_2 \\
\Rightarrow v &= 2a(18-t) \\
&= 36a - at.
\end{aligned}$$

Setting the v expressions equal,

$$\begin{aligned}
\Rightarrow at &= 36a - 2at \\
\Rightarrow 3at &= 36a \\
\Rightarrow t &= 12 \\
\Rightarrow v &= at \\
&= 12a. \\
s_1 &= \left(\frac{u_1+v_1}{2}\right) t_1 \\
&= 72a. \\
s_2 &= \left(\frac{u_2+v_2}{2}\right) t_2 \\
&= 36a \\
\Rightarrow \text{Total Distance} &= s_1 + s_2 \\
&= 108a.
\end{aligned}$$

20.

Particle P	Particle Q	Extra Equations
$u_P = 2u$	$u_Q = 3u$	~~$s_P + s_Q = d$~~
$v_P =$	$v_Q =$	~~$t_P = t_Q$~~
$a_P = a$	$a_Q = -a$	
$s_P = s$	$s_Q = d - s$	
$t_P = t$	$t_Q = t$	

$$\begin{aligned}
s_P &= u_P t_P + \frac{1}{2} a_P t_P^2 \\
\Rightarrow s &= 2ut + \frac{1}{2} at^2. \\
s_Q &= u_Q t_Q + \frac{1}{2} a_Q t_Q^2 \\
\Rightarrow d - s &= 3ut - \frac{1}{2} at^2 \\
\Rightarrow -s &= -d + 3ut - \frac{1}{2} at^2 \\
\Rightarrow s &= d - 3ut + \frac{1}{2} at^2.
\end{aligned}$$

Letting the s expressions be equal,

$$\begin{aligned}
d - 3ut + \frac{1}{2} at^2 &= 2ut + \frac{1}{2} at^2 \\
\Rightarrow d &= 5ut \\
\Rightarrow \frac{d}{5u} &= t.
\end{aligned}$$

1.15 Revision

As Linear Motion is almost unchanged from the old syllabus, doing revision questions written by me is unnecessary; students should instead be practicing exam questions, of which there are decades worth.

On the 2023 exam, Linear Motion appeared in Question 5 (b). On the sample paper Linear Motion appeared in Question 3 (b). Before that, Linear Motion appeared in Question 1 of the exam paper. The list below shows the Linear Motion exam questions (post 1996) that students shouldn't attempt at this point, and why.

2020 Q1 (b) (ii) (requires Collisions)
2019 Q1 (a) (requires Connected Particles)
2018 Q1 (a) (requires Connected Particles)
2017 Q1 (b) (requires Circular Motion)
2014 Q1 (b) (requires Work, Energy & Power)
2005 Q1 (b) (requires Connected Particles)
2004 Q1 (b) (requires Work, Energy & Power)
1999 Q1 (a) (requires Work, Energy & Power)

Solutions to these questions are available on examination.ie/exammaterialarchive/ or at brendanwilliamson.ie/pastpapers.

2. Connected Particles

2.1 Newton's Three Laws of Motion

In this chapter, rather than looking at acceleration, velocity and position in isolation we will look at the underlying physics that causes objects to acceleration. To do this, we will start with a study of **Newton's Laws of Motion**.

Definition 2.1 — Newton's Laws of Motion.

- **First Law**: A body will remain in a state of rest or at constant velocity in a straight line, unless it is acted on by an external force.
- **Second Law**: The change in momentum per unit time is proportional to the force applied, and takes place in the direction of the force.
- **Third Law**: To every action there is an equal and opposite reaction.

Note 2.2 The first law states that objects don't start or stop moving by themselves. If an object slows down, speeds up, or changes direction, some force acted on it. This may be an impact with another object, friction, gravity, or something else.

Note 2.3 Regarding the second law, momentum is mass$\times$velocity, so change in momentum per unit time is

$$\begin{aligned}\frac{mv - mu}{t} &= m\frac{v-u}{t} \\ &= ma.\end{aligned}$$

So in mathematical terms, the second law states that $F \propto ma$, and since acceleration and force are vectors, they act in the same direction. We can change the proportionality to equality by noting that $F = kma$, and $1\text{ N} = 1\text{ kg m/s}^2$ and so $1 = k(1)(1)$, so $k = 1$ and $F = ma$. In reality, when inventing the Newton measurement we chose its size so that the constant of proportionality $k = 1$.

Note 2.4 The third law is referring to phenomena like recoil from firing a gun or why you lean forward when pushing something. When a force is applied in one direction, a force of equal magnitude is applied in the opposite direction. For example, if you push someone on an ice rink, you will get pushed in the opposite direction with the same force. There will be a regular appearance of equal and opposite forces in this chapter which will gradually cement your understanding of the third law.

Collectively, the most important point to take from Newton's Laws is that only force can cause acceleration, and it does so according to the equation $F = ma$. Therefore when we want to calculate the acceleration of something, we look at the forces acting on it and go from there.

2.2 Sources of Forces and Basic Problems

One of the most important skills we will learn in this chapter is figuring out where forces are coming from and how to calculate them.

Example 2.5

- A car moving along a straight road will experience a force from the engine. There will also be resisting forces from air friction, and maybe the brakes.
- An object hanging from a rope experiences a **tensile force** from the rope, pulling it upwards (this force is sometimes referred to as the tension in the rope). It also experiences a **gravitational force** pulling it down. Also, there is a tensile force of equal magnitude exerted on the other end of the rope (we will see more of this later).
- An object sitting on a horizontal table experiences what is called a **normal reaction** exerted on it by the table. The object is being pulled down by gravity, but is not moving. This is because there is an equal force pushing it up, this is a force coming from the surface of the table. You can also think of this as an application of Newton's Third Law; the table is pushing the object in the opposite direction the object is pushing down on the table, but we are only interested in the object.
- An object falling from the sky into the ground will experience a **resistance force** when it hits the ground. If instead the object falls into water, it's easier to imagine the deceleration of the object once it hits the water.
- An object sliding along a rough surface will slow down. This is because of the **frictional force** between the object and the surface.

Out of all of theses forces given there is only one that we can always calculate directly.

Rule 2.6 If an object has mass m, the force due to gravity acts directly downwards and is of magnitude mg N where $g = 9.8$.

Note 2.7 This should stand to reason. In Section 1.3 when we discussed freefall problems, we ignored other forces like air resistance so that objects in freefall only experienced gravitational force, and therefore had acceleration g irrespective of their mass. The gravitational force being of magnitude mg N then follows directly from $F = ma$.

As forces are vectors, in one dimension they can be given a sign to indicate what direction they're acting in. They can also be added to give a resultant force.

Example 2.8 A 1,000 kg car is travelling along a straight road. Its engine provides a tractive force of 4,000 N, and there is air resistance of 1,000 N. What is the acceleration of the car?

If we consider the direction of the car to be the positive $\vec{i}$ direction, then the traction of the car is a force of $4,000\vec{i}$ N and the air resistance is a force $-1,000\vec{i}$ N, so that there is a net overall force of $3,000\vec{i}$ N acting on the car (called the **resultant force**). Thus

$$\begin{aligned} F &= ma \\ \Rightarrow 3000 &= 1000a \\ \Rightarrow 3 \text{ m/s}^2 &= a. \end{aligned}$$

The term resistance will be used to refer to forces which can only cause deceleration.

Example 2.9 If a 100 g rock, falling vertically downwards, hits soft earth at a velocity of 100 m/s and stops 0.2 m into the ground, what is the deceleration of the rock? Therefore what is the resistance of the earth? Ignore the effects of gravity after the rock hits the earth.

We can actually set up a UVAST array to find a. With the downward direction positive,

$$\begin{aligned} u &= 100 \\ v &= 0 \\ a &= \\ s &= 0.2 \\ t &= \end{aligned}$$

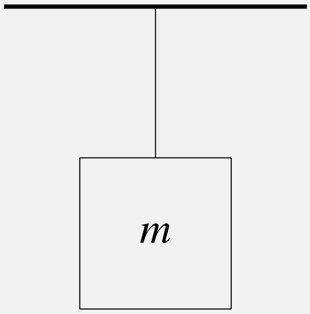
Then

$$\begin{aligned} v^2 &= u^2 + 2as \\ \Rightarrow 0 &= 10000 + 0.4a \\ \Rightarrow -25,000 \text{ m/s}^2 &= a. \end{aligned}$$

Note a is negative because the resistance force can be thought to be pushing the rock upwards. So the deceleration of the rock is $25,000$ m/s^2 and the resistance force is $F = 0.1 \times 25,000 = 2,500N$.

Example 2.10 An object is hanging from a rope attached to the ceiling. The object is not moving, and the tension in the rope is 49 N. What is the mass of the object?

Say the object is of mass m. Then the following diagram illustrates the situation.



There are two forces acting on this object: the tension in the rope, which is acting upwards and is of size 49 N, and gravity, which is acting downwards and is of size mg N. There is therefore a resultant force of $49 - mg$ N acting upwards, but as the object is not moving acceleration is 0. Therefore

$$\begin{aligned} F &= ma \\ \Rightarrow 49 - mg &= 0 \\ \Rightarrow 49 &= mg \\ \Rightarrow \frac{49}{g} &= m \\ \Rightarrow 5 \text{ kg} &= m. \end{aligned}$$

We could have simplified things by saying that if the object is not moving the upwards forces equal the downwards forces and therefore $49 = mg$.

Note 2.11 As our first example of Newton's Third Law, if the rope is exerting a force of 49 N on the hanging object, it is also exerting a force of 49 N on the ceiling, attempting to push it downwards. The resistance forces in the ceiling are complex but with a strong enough force this could cause the ceiling to collapse!

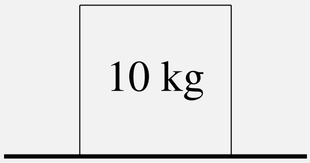
The tension in the rope is not only equal at both ends, meaning it exerts the same magnitude force on the objects tied to both ends, but at all points along the rope.

So far we have seen a few different types of forces, namely tension, resistance, gravity and direct forces such as the engine of a car. We will now consider only two more forces which are actually related, normal reactions and friction.

Normal reactions are the type of forces we talked about when considering the force that kept an object on a table stationary. They occur when an object is pressing on another object, usually due to gravity. They are called normal reactions because they act **perpendicular** to the surface they originate from.

Example 2.12 An object of mass 10 kg is resting on a flat horizontal table. Find the normal reaction between the object and the table.

The following diagram illustrates the situation.



There are two forces acting on this object: the normal reaction, which is acting upwards and is of unknown size R, and gravity, which is acting downwards and is of size $10g = 98$ N. As the object is not moving the upwards forces equal the downwards forces and so $R = 98$ N.

Note 2.13 The normal reaction can actually be thought of as coming from the table as it consciously tries to keep the 10 kg object aloft. As we have seen, on a flat horizontal table with no other forces the table has to exert a force equal to the gravitational force of the object. Therefore a heavier object places the table under more strain, and if we were on the moon the gravititaional force would be a lot weaker, and the materials of the table would be under a lot less strain! The object itself is applying the same force of the same magnitude to the table in the downwards, attempting to push it downwards. With weak enough table legs it would be successful.

Friction is closely related to normal reactions. If an object is heavier, and the surface it is resting on is exerting a larger normal reaction to prop it up, the object will be more resistant to movement, as it is pressed tighter against the surface and so causing more friction. So a larger normal reaction causes a larger frictional force when movement is attempted.

Rule 2.14 Given two objects in contact and in motion, the frictional force F between these two object is (approximately) proportional to the normal reaction R between them, with constant of proportionality μ, i.e. $F = \mu R$.

The frictional force acts in the opposite direction to movement and attempted movement, and its magnitude is only ever large enough to stop movement. So if two objects are touching but not moving and a frictional force of 30 N can be applied but the force attempting movement is 20 N, then the frictional force will only be 20 N in the opposite direction, cancelling out the other force and stopping movement.

Note 2.15 In practice, μ depends on the nature of the two surfaces in contact, but not their size or shape. Two blocks made of identical polished wood on the same marble surface would have the same coefficient of friction, even if they were of different sizes.

Note 2.16 Some problems will refer to a surface or an object as being **smooth**, which implies that there is no friction (i.e. that $\mu = 0$). On the other hand, if an object or surface is **rough** then we can be sure there is friction.

Example 2.17 A block of stone with a mass of 1,000 kg is resting on horizontal ground. The coefficient of friction between the block and the ground is $\mu = \frac{3}{4}$. A force of 10,000 N is applied to the block acting parallel to the ground (i.e. pushing the stone horizontally). Find the acceleration of the block.

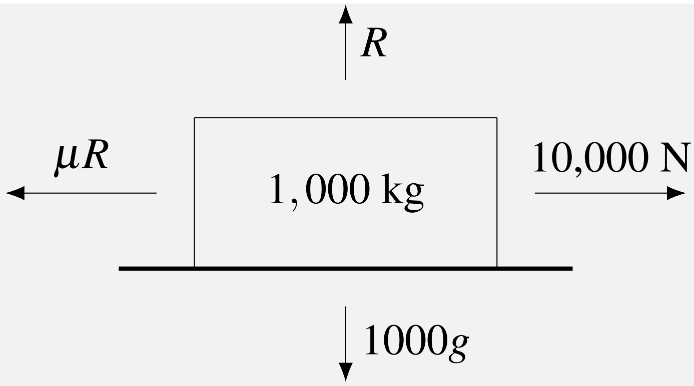


Figure 2.1

In Figure 2.1 we have all the forces acting on the block: gravity, the normal reaction between the ground and the block, the 10,000 N force applied to the stone, and the frictional force acting in the opposite direction to it. Since there is no movement in the vertical direction,

$$\begin{aligned} R &= 1000g \\ &= 9,800 \text{ N} \\ \Rightarrow F &= \mu R \\ &= \frac{3}{4}(9800) \\ &= 7,350 \text{ N}. \end{aligned}$$

Thus the overall force acting on the stone is $10,000 - 7,350 = 2,650$ N rightwards. Thus the acceleration of the stone satisfies

$$\begin{aligned} F &= ma \\ \Rightarrow 2650 &= 1000a \\ \Rightarrow 2.65 \text{ m/s}^2 &= a. \end{aligned}$$

Note 2.18 If we had instead pushed the stone with a force of 5,000 N the block would not move as this is less than the 7,350 N frictional force that we calculated. However in this case the frictional force applied would only equal 5,000 N, just enough to resist the motion. The maximum amount of friction that can be put into play, always equal to μR and in this case 7,350 N, is called the **limiting friction**.

Question 2.19 A block of stone with a mass of 300 kg is resting on horizontal ground. The coefficient of friction between the block and the ground is $\mu = 0.7$. A pack of sled dogs can be attached to the stone independently, and each one can provide a horizontal pulling force of 600 N. How many dogs need to be attached to the block so that it moves, and what is the resultant acceleration? How long does it take the dogs to move the block 100 m?

2.3 More Advanced Problems 1

The problems that we will see in this section involve a combination of flat frictional surfaces (although we will sometimes ignore friction) with objects connected by string and pulleys.

Forces coming from engines and (non-friction) resistance forces are important in other topics such as Work, Energy & Power, and Differential Equations. However from here we will only ever deal with four forces.

Note 2.20 In typical Connected Particles questions, forces from engines and resistance forces are exceedingly rare. Forces instead usually come in four types, friction, normal reaction, gravity and tension. They can be remembered using the mnemonic FRGT, as in "don't F(o)RG(e)T your forces!", and appear in the following situations.

Friction	(when objects are touching and the surface is rough)
Reaction	(when objects are touching)
Gravity	(always)
Tenstion	(when there's a rope)

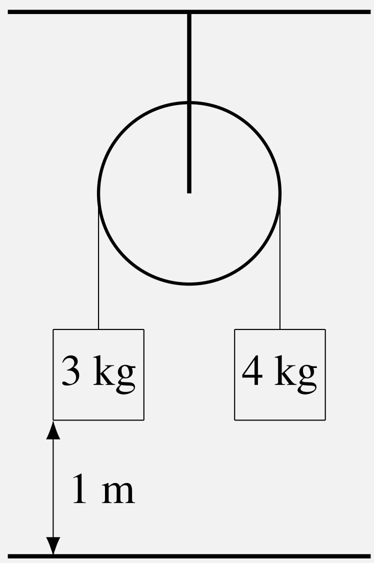
Rule 2.21 When solving problems in Connected Particles, we complete the following steps.

1. Draw all forces acting on each object, as well as the direction of acceleration of the object. If necessary resolve them along two perpendicular axes.
2. Use the equation $F = ma$ to set up simultaneous equations, one for each dimension of each object where there are force or acceleration vecotrs.
3. Solve the simultaneous equations to find the variable(s) asked for in the question.

Note 2.22 We will talk about resolving forces perpendicularly in Section 2.4.

Note 2.23 As these problems result in objects with constant acceleration, after solving for the acceleration of an object we may finish by solving a basic Linear Motion problem.

Example 2.24 Two particles are connected by a taut inelastic string resting on a smooth pulley above a table as shown below. The 4 kg particle is 1 m above the table.



(a) What is the common acceleration of the particles?

(b) What is the tension in the string?

(c) How long does it take for the 4 kg particle to hit the table?

(d) How much higher does the 3 kg mass travel after the 4 kg mass hits the table?

(a) The figure below shows all forces acting on each object, and the direction of their common acceleration.

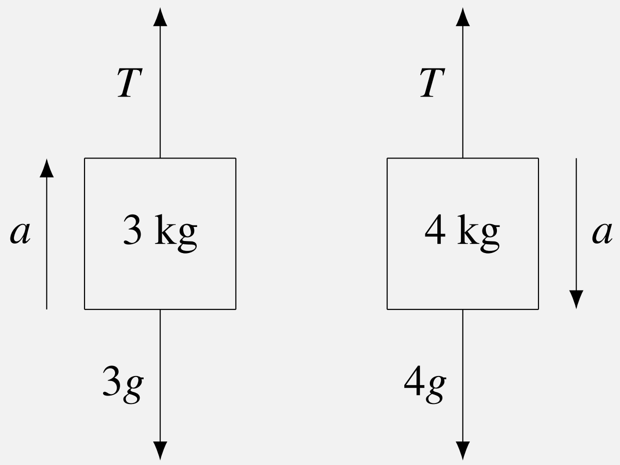


Figure 2.2

Note that the tensile forces acting on both objects is T; we discussed this as an application of Newton's Third Law in Note 2.11.

Applying the equation $F = ma$ to the forces and accelerations of each object, with the direction of the acceleration being considered the positive direction in each case we have that

$$T - 3g = 3a,$$
$$4g - T = 4a.$$

T and a are unknown, but $g = 9.8$ is not. Adding the equations to eliminate T we get

$$\Rightarrow g = 7a$$
$$\Rightarrow \frac{g}{7} = a.$$

(b) Once we know a, getting T is almost immediate.

$$\begin{aligned} T - 3g &= 3a \\ \Rightarrow T &= 3g + 3\left(\frac{g}{7}\right) \\ &= \frac{24g}{7}. \end{aligned}$$

(c) To see how long it takes the 4 kg particle to reach the ground, as the acceleration is constant this is a Linear Motion problem. In particular, taking the downwards direction as positive we have

$$\begin{aligned} u &= 0 \\ v &= \\ a &= \frac{g}{7} \\ s &= 1 \\ t &= \end{aligned}$$

and so

$$\begin{aligned} s &= ut + \frac{1}{2}at^2 \\ \Rightarrow 1 &= \frac{1}{2}\frac{g}{7}t^2 \\ \Rightarrow \frac{14}{g} &= t^2 \\ \Rightarrow \sqrt{\frac{14}{g}} &= t. \end{aligned}$$

(d) To calculate how high the 3 kg particle travels after the 4 kg particle hits the table, note first that there is no tension acting on it; the string is **slack**. So the only force acting on it is gravity; it's in freefall. So we again treat it as a Linear Motion problem, setting $a = -g$, $v = 0$. For our third quantity, we go back to the previous Linear Motion problem and calculate the velocity of the 4 kg particle at the time when it hit the table. Calculating that

$$\begin{aligned} v &= u + at \\ &= 0 + \frac{g}{7}\sqrt{\frac{14}{g}} \\ &= \sqrt{\frac{2g}{7}}, \end{aligned}$$

we can let

$$\begin{aligned} u &= \sqrt{\frac{2g}{7}} \\ v &= 0 \\ a &= -g \\ s &= \\ t &= \end{aligned}$$

Then

$$\begin{aligned} v^2 &= u^2 + 2as \\ \Rightarrow 0^2 &= \sqrt{\frac{2g}{7}}^2 - 2gs \\ \Rightarrow 2gs &= \frac{2g}{7} \\ \Rightarrow s &= \frac{1}{7} \text{ m.} \end{aligned}$$

Note 2.25 In Example 2.24, only the ratio between the masses of the particles is important in calculating a. You can see this by replacing 3 and 4 with $3M$ and $4M$ respectively, or also by noticing that the distance, velocity and acceleration vectors are independent of the kg measurement. This will be a common sight in exams, and it will in fact be rarer to be given numerical mass measurements.

Example 2.26 An object of mass $2M$ is placed on a table. It is connected to a second particle of mass M by a taut inelastic string passing over a smooth pulley hanging over the edge of the table, as in the diagram below.

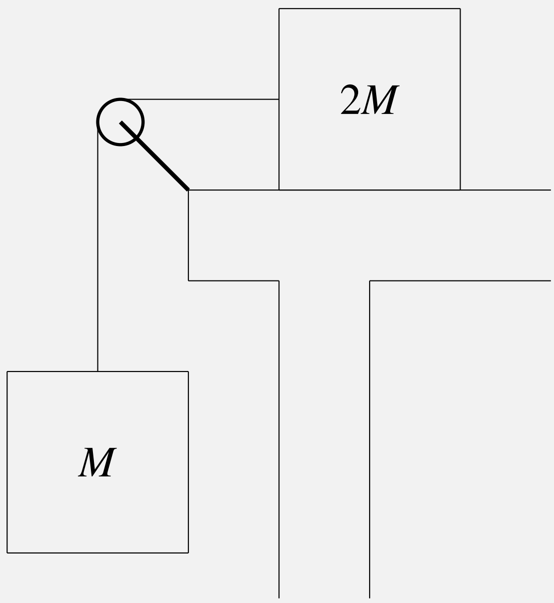


Figure 2.3

(a) Assume first that the table is smooth. What is the common acceleration of the particles?

(b) What is the common acceleration of the particles if the table is rough, with a coefficient of friction between the particle of mass $2M$ and the table of $\mu = \frac{1}{5}$?

(c) What is the smallest value of μ such that the acceleration of the particles is zero?

The following are the forces acting on each object in the general case of unknown μ, which I've drawn once for all three problems.

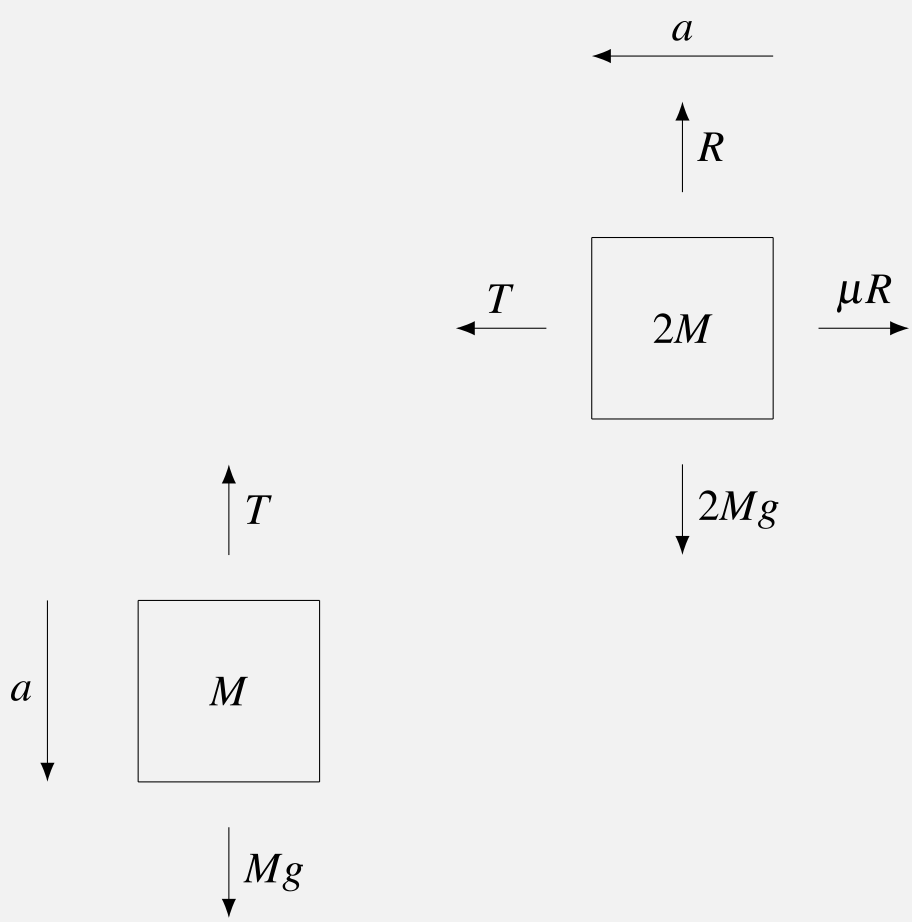


Figure 2.4

(a) Here assume $\mu = 0$. The vertical forces on the $2M$ object give us $R = 2Mg$. From the horizontal forces we get only a force of T to the left, and so $T = 2Ma$. From the M object we deduce that $Mg - T = Ma$, giving us the system of simultaneous equations

$$\begin{aligned} R &= 2Mg, \\ T &= 2Ma, \\ Mg - T &= Ma. \end{aligned}$$

Here again R, a and T are unknowns in this system, g is not and M is not either, because it is unimportant (and actually unknowable). We don't actually care about R, only a, so ignoring the first equation and substituting the second into the third we get

$$\begin{aligned} Mg - 2Ma &= Ma \\ \Rightarrow Mg &= 3Ma \\ \Rightarrow \frac{g}{3} &= a. \end{aligned}$$

(b) If $\mu = \frac{1}{5}$, then there is an additional frictional force of $\mu R = \frac{R}{5}$ acting to the right on the

$2M$ object, so that our equations are slightly altered to

$$\begin{aligned} R &= 2Mg, \\ T - \frac{R}{5} &= 2Ma, \\ Mg - T &= Ma. \end{aligned}$$

As M and g are not considered unknowns we can consider R as "solved for". Substituting its value into the second equation, then adding them we get

$$\begin{aligned} T - \frac{2Mg}{5} &= 2Ma, \\ Mg - T &= Ma \\ \Rightarrow \frac{3Mg}{5} &= 3Ma \\ \Rightarrow \frac{g}{5} &= a. \end{aligned}$$

(c) To see the smallest μ to guarantee inertia, assume that μ is just large enough that the limiting friction is applied to the $2M$ object but $a = 0$. Then our equations become

$$\begin{aligned} R &= 2Mg, \\ T &= \mu R, \\ Mg &= T. \end{aligned}$$

In this case we can consider R and T as "known" or "solved for". Substituting them into the second equation gives us

$$\begin{aligned} Mg &= \mu(2Mg) \\ \Rightarrow 1 &= 2\mu \\ \Rightarrow \frac{1}{2} &= \mu. \end{aligned}$$

There are many interesting things to consider with this example, which will apply to almost all problems in this chapter.

Note 2.27 When we construct the simultaneous equations when solving Connected Particles problems, it's useful to know which letters are really variables.

- As stated before $g = 9.8$ should not be considered a variable.
- If all masses are given as multiples of M then M should not be considered a variable either as it is impossible to solve for unless other information is given (we will consider some unique exam questions that are exceptions to this, for example when some masses are given as numbers and some are not, see Question 2.54 later).
- Unknown accelerations, tensions, normal reactions, and coefficients of friction μ should be considered variables. In the case of masses given as multiples of M, after solving the simultaneous equations accelerations should be multiples of g, and forces should be multiples of Mg. If this is not the case you have made an algebra mistake somewhere. This also means that finding $T = \frac{9}{7}Mg$ should be considered "solving for T", as that's the best you can do.

Note 2.28 When solving for accelerations, as there are no engines or resistance forces other than friction, when all objects start from rest all acceleration variables should be smaller than g. Therefore if you find $a = \frac{11}{9}g$ for example, you have made a mistake somewhere.

Note 2.29 Notice a contrast between part (a) and parts (b) & (c) of Example 2.26. In all parts our diagrams and simultaneous equations were similar enough. However when solving, in all cases we knew R straight away but with no friction in (a) we didn't care about R, whereas we did in parts (b) and (c). This is a common theme in problems with normal reactions. We will almost always be able to find them straight away, but unless the question asks for them directly we will only care about them if there is friction in the problem.

Question 2.30 A particle of mass M hangs freely as it is attached to a taut string. The string passes over a light, smooth pulley and is attached at its other end by a particle of mass $2M$ which lies on a flat, horizontal table. Another string is attached to the other side of the $2M$ particle. It passes over another light, smooth pulley and is attached at its other end to a particle of mass $3M$ which hangs freely, as in the following diagram.

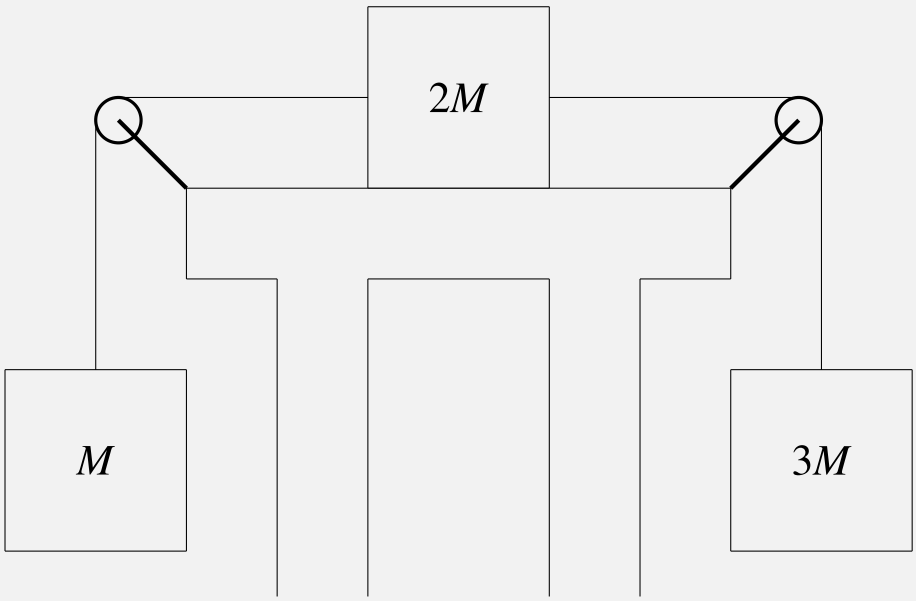


Figure 2.5

What is the common acceleration of the objects

(a) if the table is smooth?

(b) if the table and $2M$ particle have a coefficient of friction of $\mu = 0.5$?

Hint: There are two strings in this system and they have different tensions, which you might call S and T.

Note 2.31 As you may have noticed at this point, strings with tension T often contribute a T term to one equation and a $-T$ term to another, making adding equations a reliable way to remove the T variable from the system of equations.

2.4 Resolving Forces

So far every force we've seen has been either horizontal or vertical. To deal with forces that are oblique we must **resolve** them into perpendicular components.

Rule 2.32 Any force can be resolved into perpendicular components by considering it as the hypotenuse in a right angled triangle with known angles and finding the other two sides using trigonometry. These two perpendicular forces are considered equivalent to the original force.

Note 2.33 For example, consider a force of 5 N acting on an object at an angle of 36.87° to the horizontal. If we want to resolve this force into its horizontal and vertical components, we draw it as the hypotenuse of a triangle with horizontal and vertical sides.

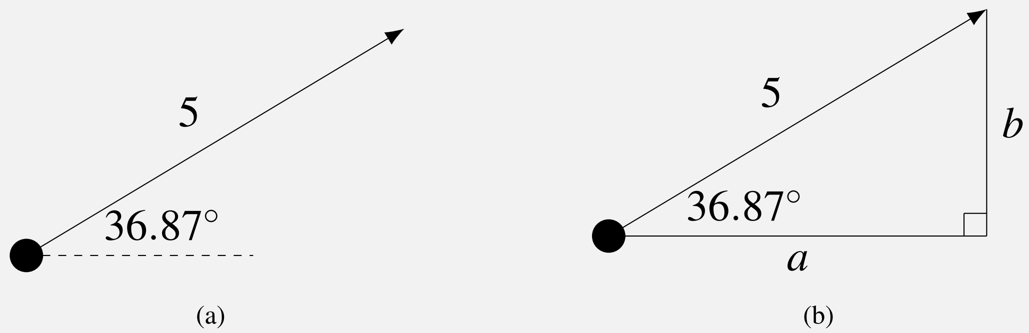


Figure 2.6

Letting the horizontal and vertical sides be a and b respectively, we can show that

$$\begin{aligned}\cos 36.87^\circ &= \frac{a}{5}\\ \Rightarrow 5\cos 36.87^\circ &= a\\ \Rightarrow 4 &= a,\\ \sin 36.87^\circ &= \frac{b}{5}\\ \Rightarrow 5\sin 36.87^\circ &= b\\ \Rightarrow 3 &= b.\end{aligned}$$

What this means is that if we pull on this object with a force of 5 N at an angle of 36.87° to the horizontal, it will have **exactly the same effect** as if we pull on it with a force of 4 N horizontally and a force of 3 N vertically.

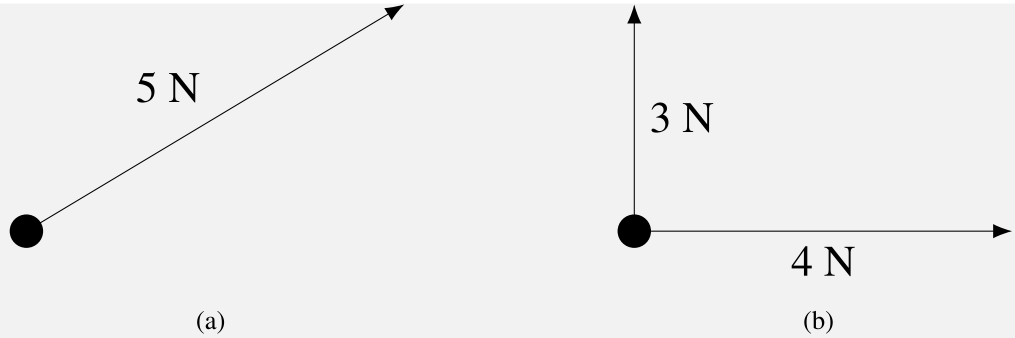


Figure 2.7

Therefore when drawing our forces acting on an object, we can swap out the 5 N for the 4 N and 3 N forces.

Note 2.34 In general, when resolving a force of h Newtons into perpendicular components using an angle of θ, one component will be $h\cos\theta$ and the other will be $h\sin\theta$.

Let's consider an altered version of Example 2.17.

Example 2.35 A block of stone with a mass of 1,000 kg is resting on horizontal ground. The coefficient of friction between the stone and the ground is $\mu = \frac{3}{4}$. A force of 10,000 N is applied to the block acting at an angle of 36.87° to the ground. Find the acceleration of the block.

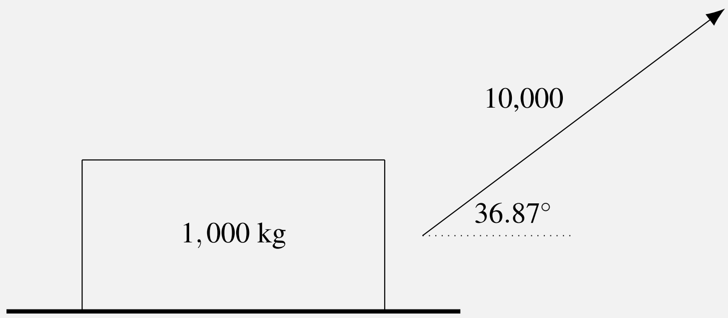


Figure 2.8

In Figure 2.9 below we have all the forces acting on the block, with the oblique force of 10,000 N resolved into horizontal and vertical components. This is necessary; remember Rule 2.21 says that forces and accelerations must be resolved along perpendicular axes, in this case the horizontal and vertical axes so that all arrows are on the same axes.

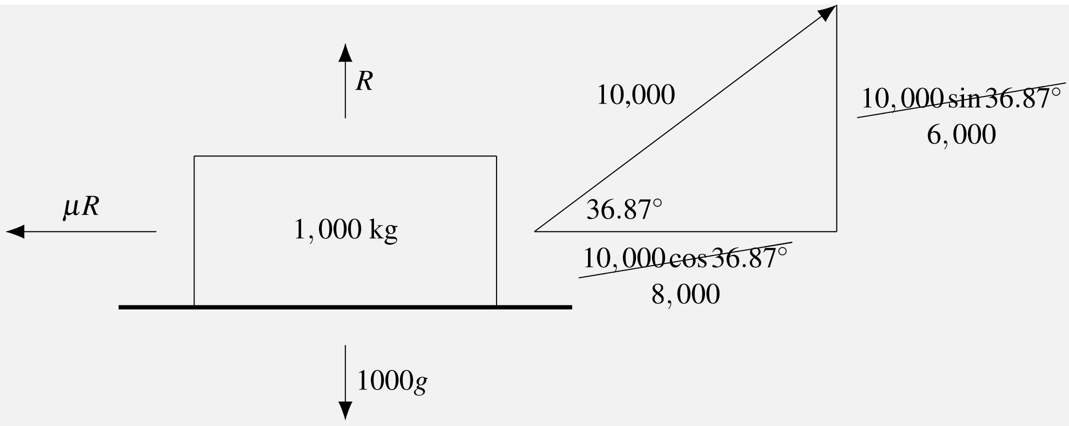


Figure 2.9

From here we can proceed as before. As there is no movement in the vertical direction,

$$R + 6000 = 1000g$$
$$\Rightarrow R = 9800 - 6000$$
$$= 3,800 \text{ N}.$$

If the block moves to the right with acceleration a,

$$F = ma$$
$$\Rightarrow 8000 - \mu R = 1000a$$
$$\Rightarrow 8000 - \frac{3}{4}(3800) = 1000a$$
$$\Rightarrow 5150 = 1000a$$
$$\Rightarrow 5.15 \text{ m/s}^2 = a.$$

Question 2.36 A block of stone with a mass of 500 kg is resting on horizontal ground. The coefficient of friction between the stone and the ground is $\mu = \frac{1}{2}$. A force of 13,000 N is applied to the stone acting at an angle of 22.62° to the horizontal. Find the acceleration of the block.

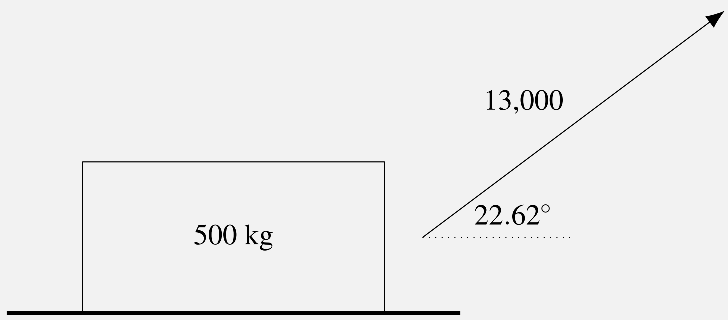


Figure 2.10

The angles given so far have been specially chosen to give "nice" magnitudes of the component forces. A more common way of doing this in Applied Maths is to not to give the angle itself, but that **tan** of the angle.

Example 2.37 A force of 10 N is applied to an object, acting at an angle of $\theta = \tan^{-1}\left(\frac{4}{3}\right)$ to the horizontal. Resolve this force into its horizontal and vertical components.

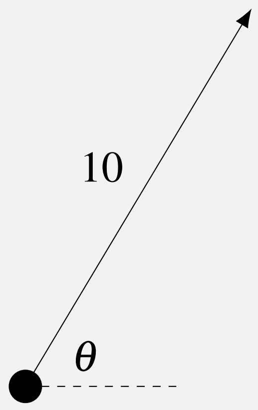


Figure 2.11

The trick is that given $\theta = \tan^{-1}\left(\frac{4}{3}\right)$ we know θ is acute and $\tan\theta = \frac{4}{3}$, meaning we can draw it inside a right-angled triangle with known opposite and adjacent (remember $\tan\theta = \frac{\text{opp}}{\text{adj}}$).

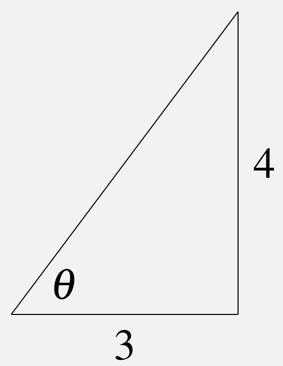


Figure 2.12

Pythagoras' Theorem can given us the hypotenuse h.

$$\begin{aligned} h^2 &= 3^2 + 4^2 \\ &= 25 \\ \Rightarrow h &= 5. \end{aligned}$$

Therefore

$$\begin{aligned} \sin\theta &= \frac{4}{5}, \\ \cos\theta &= \frac{3}{5}, \end{aligned}$$

which we found without ever finding θ itself. Then, resolving the force into its horizontal and vertical components,

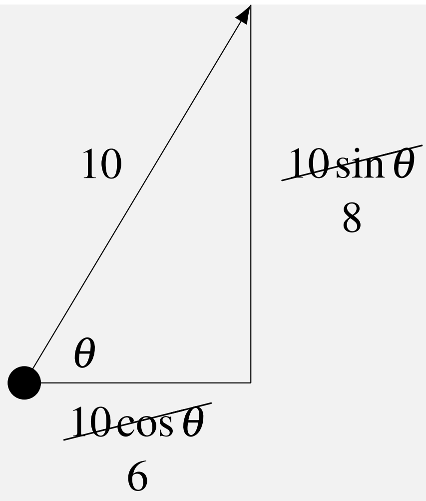


Figure 2.13

Note 2.38 It may be tempting to calculate θ directly and use that decimal to calculate $\cos\theta$ and $\sin\theta$, but that doesn't work as soon as surds are involved.

Question 2.39 A force of 5 N is applied to an object, acting at an angle of $\theta = \tan^{-1}\left(\frac{1}{2}\right)$ to the horizontal. Resolve this force into its horizontal and vertical components.Give your answers in surd form.

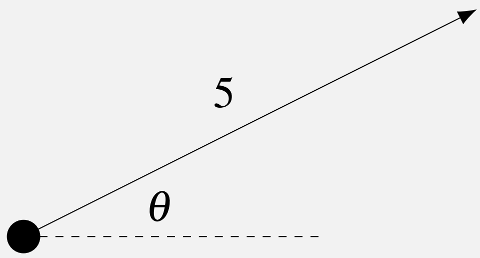


Figure 2.14

2.5 More Advanced Problems 2

So far all the problems we have seen involved pulleys, strings and flat surfaces. Now consider a particle on an angled surface.

Example 2.40 A particle of mass M lies on an immovable plane that is angled at $\alpha = \tan^{-1}\left(\frac{1}{2}\right)$ to the horizontal.

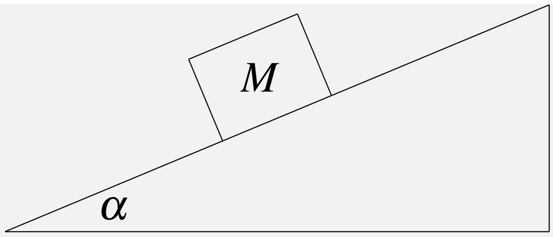


Figure 2.15

What is its acceleration if

(a) The plane is smooth?

(b) The coefficient of friction between the plane and the block is $\mu = \frac{1}{5}$?

The forces acting on the particle in the case of general μ, as well as the direction of its acceleration are as follows.

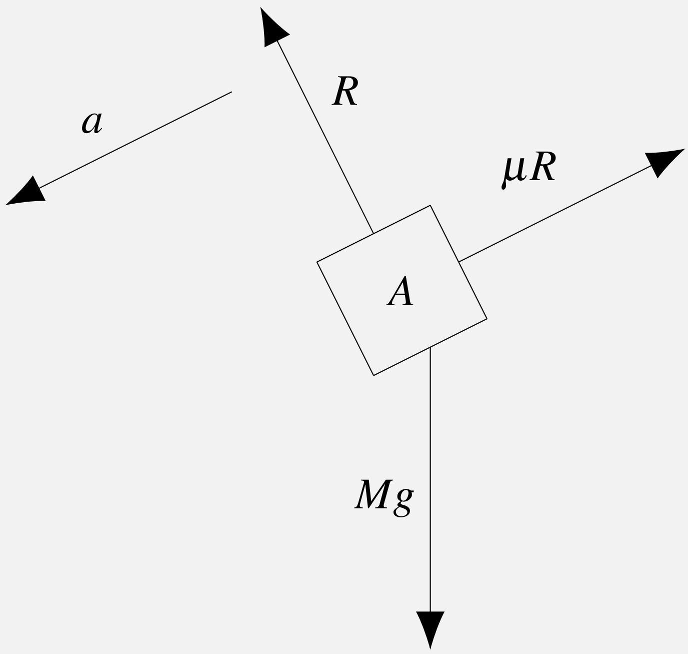


Figure 2.16

Notice that friction acts opposite to the direction of motion, and the normal reaction is normal (i.e. perpendicular) to the surface the particle is resting on.

We need to resolve these forces along perpendicular axes. Resolving the μR and R forces (as well as a) horizontally and vertically is a lot more work than if we just resolved Mg to be parallel and perpendicular to the plane, giving us the following resolved forces. Use whatever version of alternate/corresponding angles etc. to convince yourself the given angle is equal to α.

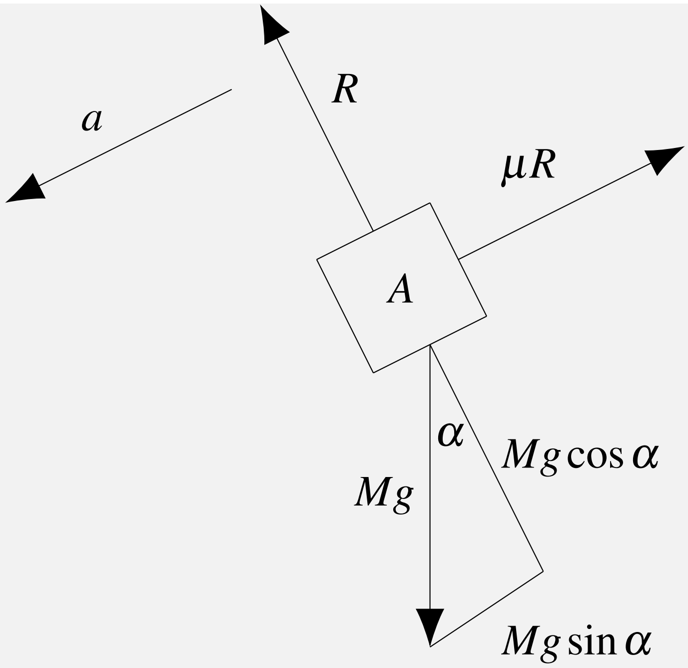


Figure 2.17

First we need to find $\cos\alpha$, $\sin\alpha$ given $\tan\alpha = \frac{1}{2}$.

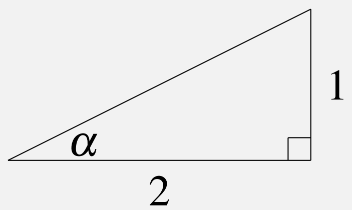


Figure 2.18

Some quick calculations give a hypotenuse of $\sqrt{5}$ so that

$$\cos\alpha = \frac{2}{\sqrt{5}},$$
$$\sin\alpha = \frac{1}{\sqrt{5}}.$$

(a) Applying $F = ma$ to the forces perpendicular and parallel to the plane give us

$$R = Mg\cos\alpha,$$
$$Mg\sin\alpha = Ma$$

which simplify to

$$R = \frac{2Mg}{\sqrt{5}},$$
$$\frac{Mg}{\sqrt{5}} = Ma$$

so that

$$\frac{g}{\sqrt{5}}\ \mathrm{m/s^2} = a.$$

See that we again found R immediately but didn't need it in the frictionless case, as stated in Note 2.29.

(b) In this case our equations are now

$$R = Mg\cos\alpha,$$
$$Mg\sin\alpha - \mu R = Ma$$

which simplify to

$$R = \frac{2Mg}{\sqrt{5}}$$
$$\frac{Mg}{\sqrt{5}} - \frac{1}{5}\frac{2Mg}{\sqrt{5}} = Ma$$

so that

$$\frac{g}{\sqrt{5}} - \frac{2g}{5\sqrt{5}} = a$$
$$\Rightarrow \frac{3g}{5\sqrt{5}}\ \mathrm{m/s^2} = a.$$

What if the plane is instead an immovable wedge and the block in Example 2.40 was attached by a string to another object hanging off the vertical side of the wedge?

Example 2.41 A particle of mass M lies on an immovable wedge that is angled at $\alpha = \tan^{-1}\left(\frac{3}{4}\right)$ to the horizontal. Attached to it is a light inelastic string, which passes over a pulley and is attached to an object, also of mass M, hanging off the vertical side of the wedge without touching it. The coefficient of friction between the angled part of the plane and the block is $\mu = \frac{1}{5}$. What is the common acceleration of the particles?

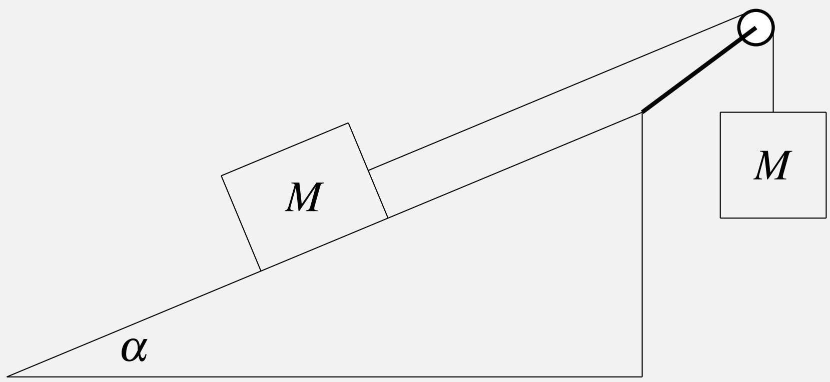


Figure 2.19

It shouldn't surprise you at this point that given $\tan\alpha = \frac{3}{4}$ we need to calculate $\cos\alpha$, $\sin\alpha$ by drawing a triangle.

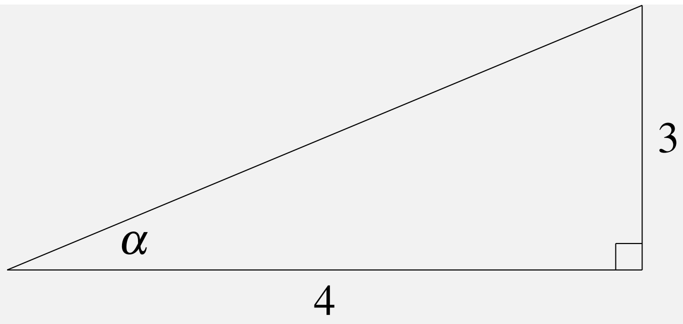


Figure 2.20

Pythagoras' Theorem quickly gives a hypotenuse of 5 so that

$$\cos\alpha = \frac{4}{5}$$
$$\sin\alpha = \frac{3}{5}.$$

The forces acting on each block, with Mg on the angled block resolved parallel and perpendicular to the plane, are given in the figure below. The common tension in the string is given by T.

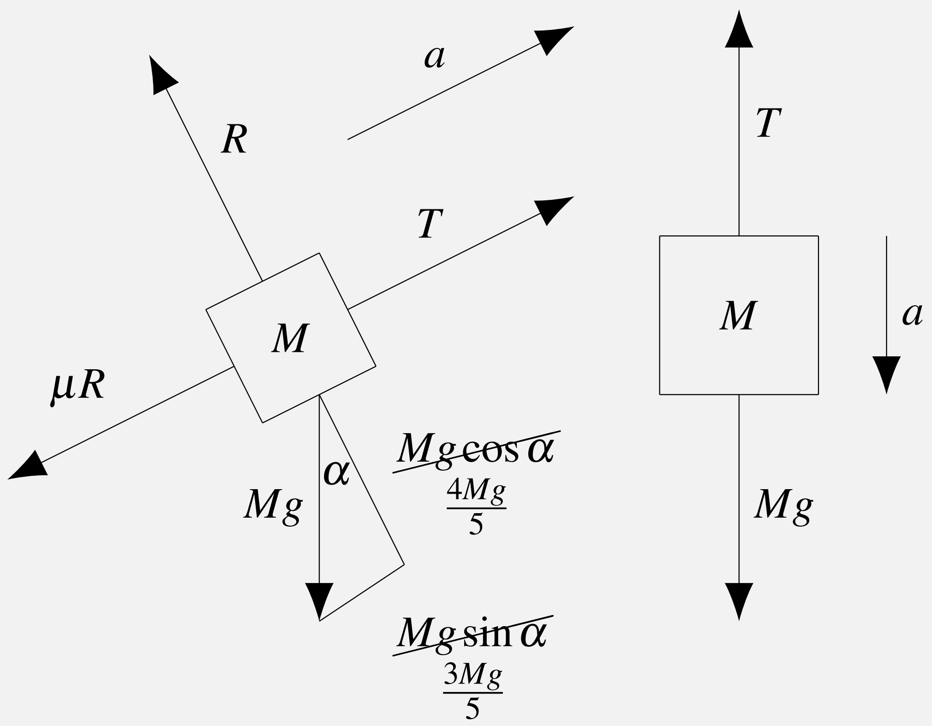


Figure 2.21

Our equations are

$$R = \frac{4Mg}{5}$$
$$T - \frac{1}{5}R - \frac{3Mg}{5} = Ma,$$
$$Mg - T = Ma.$$

Replacing R in the second equation and simplifying gives

$$T - \frac{1}{5}\frac{4Mg}{5} - \frac{3Mg}{5} = Ma$$
$$\Rightarrow T - \frac{19Mg}{25} = Ma.$$

Adding the two remaining equations

$$T - \frac{19Mg}{25} = Ma,$$
$$Mg - T = Ma$$

gives

$$\frac{6Mg}{25} = 2Ma$$
$$\Rightarrow \frac{3g}{25}\ \text{m/s}^2 = a.$$

Question 2.42 A particle of mass M lies on an immovable wedge that is angled at 30° to the horizontal. Attached to it is a light inelastic string, which passes over a pulley and is attached to an object of mass $2M$, hanging off the vertical side of the wedge without touching it. The coefficient of friction between the angled part of the plane and the block is $\mu = \frac{1}{4}$. What is the common acceleration of the particles?

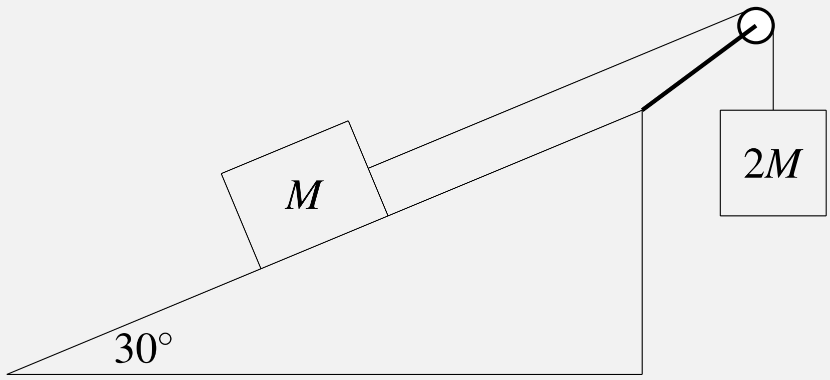


Figure 2.22

Question 2.43 In Question 2.42, would the common acceleration of the particles increase or decrease if

(a) the angle of 30° increased?

(b) μ increased?

(c) M increased?

(d) the mass of only the free hanging particle increased?

2.6 More Advanced Problems 3

The problems in this section are similar to those covered previously. The main difference is that there may be more than one acceleration, or it may not be obvious in what direction the objects are moving.

First, consider what would happen if we adjusted the setup of Example 2.41 so that the vertical side was now also angled.

Example 2.44 A smooth immovable wedge has two smooth particles, one on each of its angled sides. They are connected by a light inelastic string that passes over a pulley, as shown in the diagram below.

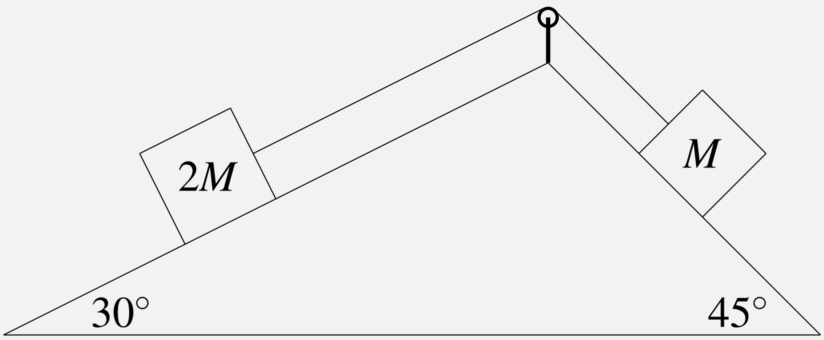


Figure 2.23

Find the common acceleration of the particles.

The figure below shows the forces acting on each particle, resolved parallel and perpendicular to their respective planes. We assume that acceleration is happening in the clockwise direction. As there is no frictional force, if we are wrong in this assumption we will get a negative answer for a, which isn't a problem.

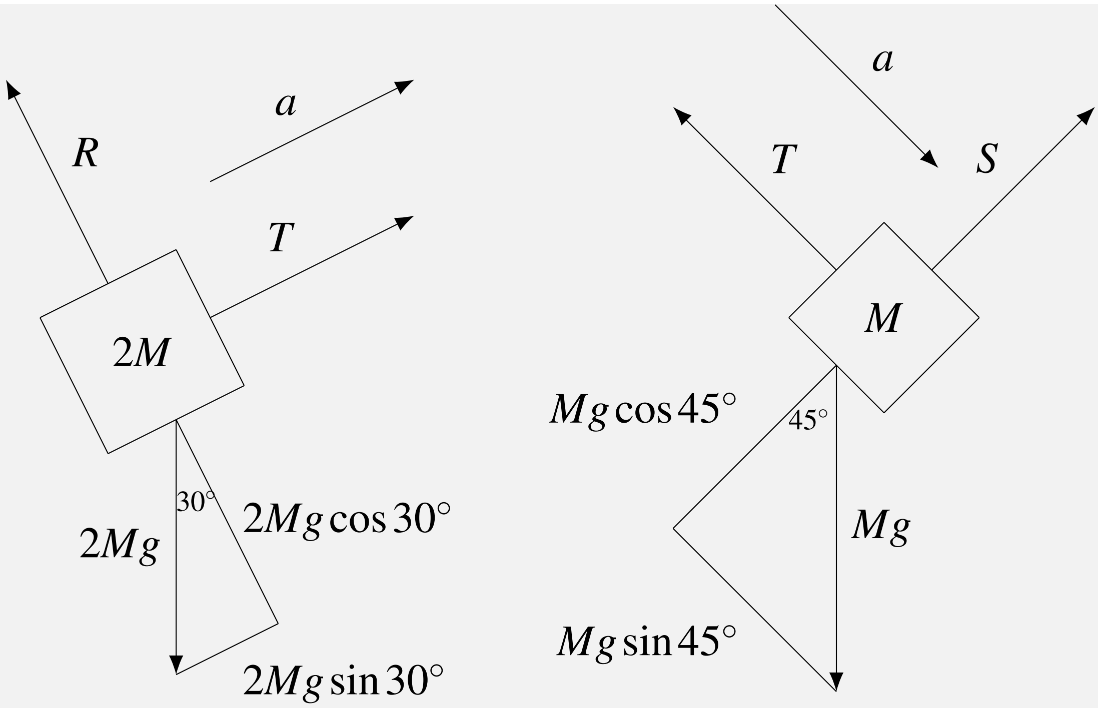


Figure 2.24

Our equations are then

$$
\begin{aligned}
R &= 2Mg\cos 30^\circ,\\
T - 2Mg\sin 30^\circ &= 2Ma,\\
S &= Mg\cos 45^\circ\\
Mg\sin 45^\circ - T &= Ma,
\end{aligned}
$$

Ignoring the equations that give us R and S, simplifying the other two to

$$
\begin{aligned}
T - Mg &= 2Ma,\\
\frac{Mg}{\sqrt{2}} - T &= Ma
\end{aligned}
$$

and adding them we get

$$
\begin{aligned}
&\frac{Mg}{\sqrt{2}} - Mg = 3Ma\\
\Rightarrow\ &\frac{1-\sqrt{2}}{\sqrt{2}}Mg = 3Ma\\
\Rightarrow\ &\frac{1-\sqrt{2}}{3\sqrt{2}}g = a.
\end{aligned}
$$

As we can see, $a < 0$ and so what actually happens is that the $2M$ particle falls, and the M particle rises, with acceleration

$$a = \frac{\sqrt{2}-1}{3\sqrt{2}}g$$
$$\approx 0.96 \text{ m/s}^2.$$

Note 2.45 If it not obvious in what direction an object will move in an exam question, one of two things should happen, depending on whether or not there is friction.

- If there is friction you will be told in what direction the object moves (see Question 2.53 in Section 2.7).That way you know in what direction the frictional force should act.
- If there is no friction then guessing the wrong direction of acceleration in inconsequential, as in Example 2.44.

Question 2.46 An smooth immovable wedge has two smooth particles, one on each of its angled sides. The angles obey $\tan\alpha = \frac{5}{12}$, $\tan\beta = \frac{4}{3}$. They are connected by a light inelastic string that passes over a pulley, as shown in the diagram below.

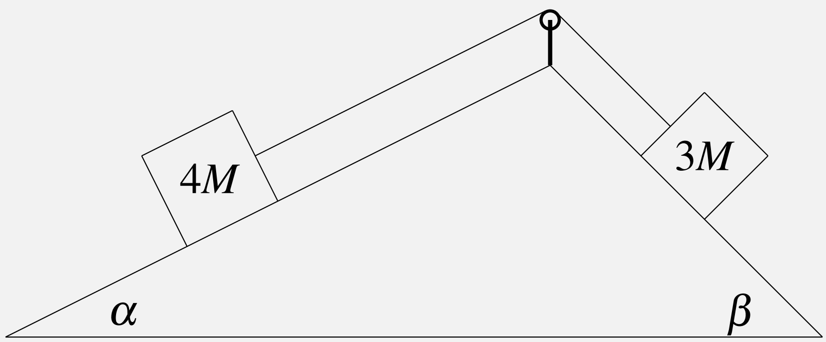


Figure 2.25

Find the common acceleration of the particles.

Example 2.47 A particle of mass M is attached to a string that passes over a light smooth pulley that is fixed to a wall, as shown in the diagram. The string then passes around another light smooth pulley that is attached to a particle of mass $2M$ that lies on a flat table. The other end of the string is fixed to the wall. The coefficient of friction between the $2M$ particle and the table is $\frac{1}{2}$.

The system is released from rest.

Find the acceleration of each particle.

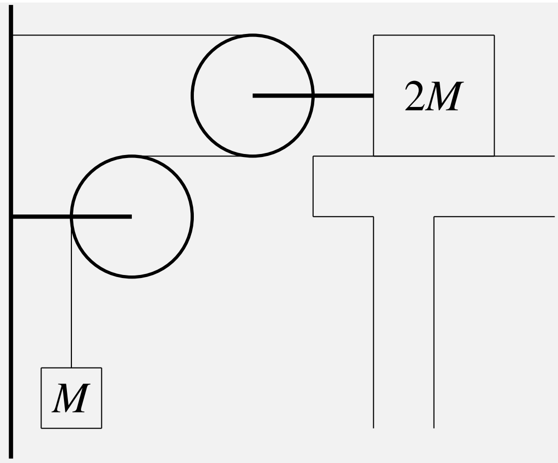


Figure 2.26

First understand that if the M particle moves down a distance d, then because of the way the rope is folded over, the $2M$ particle will move a distance of $\frac{d}{2}$ to the left. Therefore if the M particle has acceleration a than the $2M$ particle has acceleration $\frac{a}{2}$.

The forces and accelerations acting on each particle are then shown below.

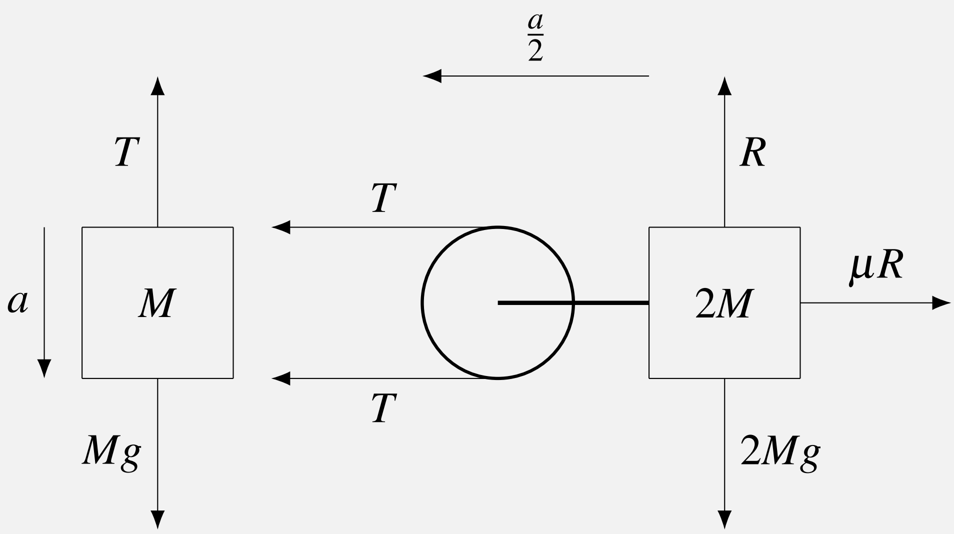


Figure 2.27

Notice that as it is the same string throughout, and therefore has the same tension throughout (as stated in Note 2.11), the $2M$ particle gets two leftwards T forces as it is being pulled up by the string in two places.

This gives us the equations

$$R = 2Mg$$
$$2T - \mu R = 2M\frac{a}{2}$$
$$Mg - T = Ma.$$

Get $T = Mg - Ma$ from the third equation, and plug this and the value for R into the second equation to get

$$\begin{aligned} 2(Mg - Ma) - \frac{1}{2}(2Mg) &= 2M\frac{a}{2} \\ \Rightarrow 2Mg - 2Ma - Mg &= Ma \\ \Rightarrow Mg &= 3Ma \\ \Rightarrow \frac{g}{3} &= a. \end{aligned}$$

So the M block has acceleration $\frac{g}{3}$ downwards and the $2M$ block has acceleration $\frac{g}{6}$ leftwards.

Question 2.48 A taut string, attached at one end to a particle of mass $5M$, passes over a fixed pulley, under a moveable pulley of mass $3M$, and is attached to the ceiling at the other end. Find the acceleration of each object, making sure to clearly give direction of acceleration.

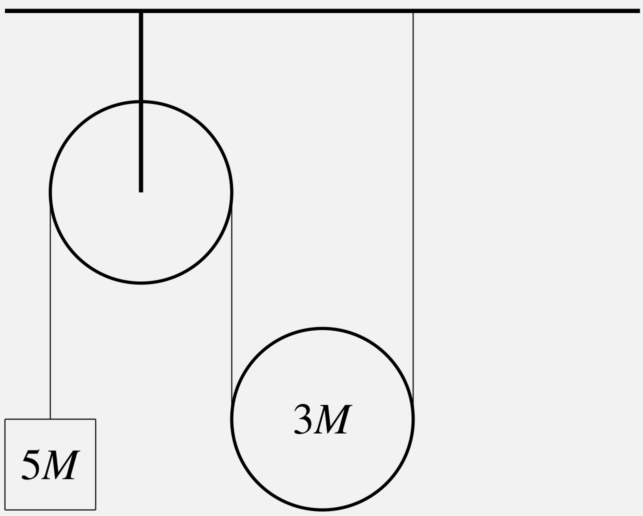


Figure 2.28

Our final example involves a single string with multiple accelerations.

Example 2.49 A string is attached on one end to a particle A of mass M. It passes over a smooth light fixed pulley, then under a smooth movable pulley B of mass $4M$, then over another light fixed pulley. It is attached at the other end to a particle C of mass $2M$. The system is released from rest. What is the acceleration of each particle A, B and C?

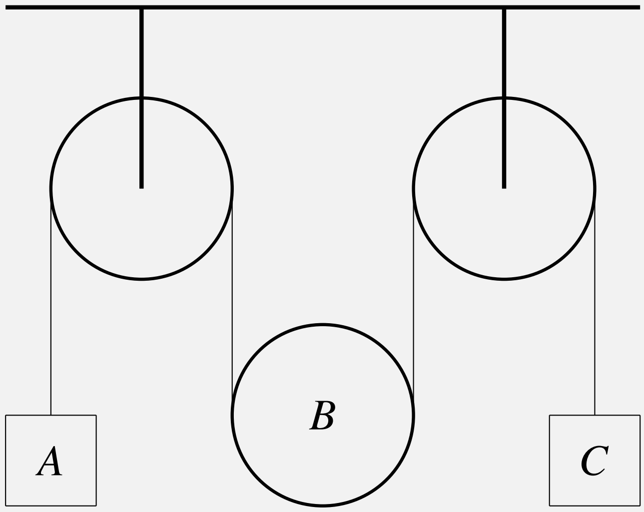


Figure 2.29

First imagine that C is kept fixed. Similarly to Example 2.47, if A moves upwards a distance d, how much will B move downwards? It will move downwards a distance $\frac{d}{2}$. By a similar logic, if A has an acceleration of a upwards then B will have an acceleration of $\frac{a}{2}$ downwards. Similarly if A is kept fixed and C has an acceleration of c upwards then B has an acceleration of $\frac{c}{2}$ downwards. Now back to the original scenario; if all particles are free to move and particles A and C have upward accelerations of a and c respectively, then particle B has an acceleration of $\frac{a+c}{2}$ downwards.

The following diagram shows the forces acting on each particle, and their accelerations.

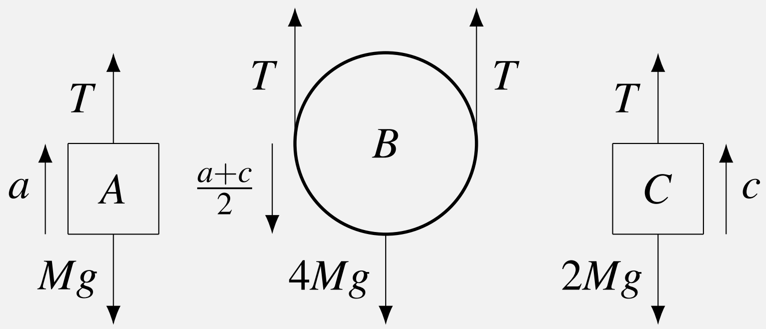


Figure 2.30

Notice that as it is the same string throughout, and therefore has the same tension throughout (as stated in Note 2.11), the particle B gets two upwards T forces as it is being pulled up by the string in two places.

This gives us the equations

$$\begin{aligned} T - Mg &= Ma \\ 4Mg - 2T &= 4M\left(\frac{a+c}{2}\right) \\ T - 2Mg &= 2Mc. \end{aligned}$$

Using the first equation to write T in terms of a and substituting it into the second and third equations to remove it from the system we get

$$\begin{aligned} T &= Mg + Ma \\ \Rightarrow 4Mg - 2(Mg + Ma) &= 2Ma + 2Mc \\ \Rightarrow 4Mg - 2Mg - 2Ma &= 2Ma + 2Mc \\ \Rightarrow 2Mg &= 4Ma + 2Mc \\ \Rightarrow g &= 2a + c, \end{aligned}$$

and

$$\begin{aligned} Mg + Ma - 2Mg &= 2Mc \\ \Rightarrow -Mg &= -Ma + Mc \\ \Rightarrow -g &= -a + 2c. \end{aligned}$$

Solving the system of equations

$$\begin{aligned} g &= 2a + c \\ -g &= -a + 2c \end{aligned}$$

gives us

$$\begin{aligned} a &= \frac{3g}{5} \\ c &= -\frac{g}{5}. \end{aligned}$$

Therefore particle A moves upwards with acceleration $\frac{3g}{5}$, and particle C moves **downwards** with acceleration $\frac{g}{5}$. Particle B moves downwards with acceleration

$$\frac{\frac{3g}{5} - \frac{g}{5}}{2} = \frac{g}{5}.$$

Note that again it was unimportant that we incorrectly assumed particle C moved upwards when the system was released from rest.

Note 2.50 When attempting questions like this, it is important to have A and C go "in the same direction" so that the acceleration of the middle pulley B is easier to construct. Assuming A and C both went down would have been an acceptable assumption in this question, but it would be unwise to initially assume that A went up and C went down, even though that was actually the case.

It may not be intuitive that with three objects in Example 2.49 that two particles can be given independent accelerations, at which point the third object has its acceleration given in terms of the other two. The following rule tells us how many acceleration variables we should have in any problem in this chapter.

Rule 2.51 For any problem involving particles, pulleys, strings, friction and/or angled surfaces, if no two movable particles are touching then

$$\text{Number of acceleration variables} = \text{Number of movable particles} - \text{Number of strings.}$$

In Example 2.49 we had 3 movable particles (not counting the pulleys fixed to the ceiling) and 1 string, and therefore needed $3-1=2$ acceleration variables to solve the problem. You can check other examples in this chapter to see how Rule 2.51 always holds true.

Question 2.52 A string is attached on one end to a particle A of mass 4 kg. It passes over a smooth light fixed pulley, then under a smooth movable pulley B of mass 2 kg, then over another light fixed pulley. It is attached at the other end to a particle C of mass 5 kg. The system is released from rest.

(a) What is the acceleration of each particle A, B and C? Make sure to give the direction of each acceleration.

(b) What is the tension in the string?

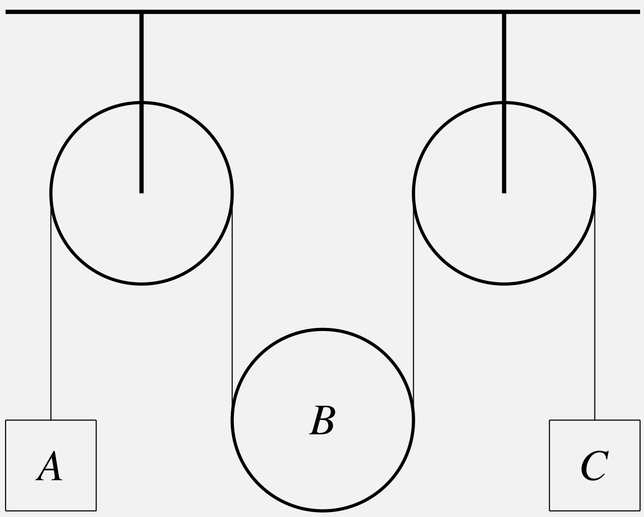
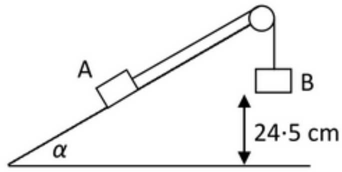


Figure 2.31

2.7 Exam Questions

Question 2.53 — 2020 Q4.

4. **(a)** A block A of mass 10m on a smooth plane inclined at an angle α with the horizontal, where $\tan \alpha = \frac{3}{4}$, is connected by a light inextensible string which passes over a smooth pulley to a second block B of mass 10m. B is 24·5 cm above an inelastic horizontal floor, as shown in the diagram.



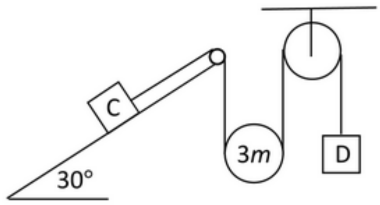
The system is released from rest.

Find

(i) the acceleration of B

(ii) the time that B remains in contact with the floor.

(b) A particle C of mass 2m rests on a rough plane which is inclined at 30° to the horizontal. The coefficient of friction between C and the plane is $\frac{\sqrt{3}}{21}$. A light inextensible string which passes under a smooth movable pulley of mass 3m connects C to a particle D of mass m, as shown in the diagram.



The system is released from rest. C moves up the plane.

(i) Show, on separate diagrams, the forces acting on the moveable pulley and on each of the masses.

(ii) Find in terms of m the tension in the string.

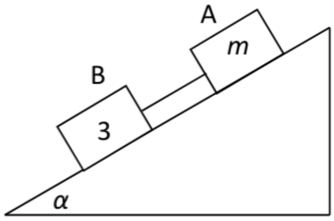
Figure 2.32

Question 2.54 — **2018 Q4.**

(a) A block A of mass m is connected by a light inextensible string to a second block B of mass 3 kg.
They slide down a rough inclined plane which makes an angle α with the horizontal where $\tan\alpha = \frac{3}{4}$.
The string remains taut in the subsequent motion.
The coefficient of friction between A and the plane is $\frac{3}{4}$.
The coefficient of friction between B and the plane is $\frac{1}{3}$.
The system is released from rest.
Find

(i) the acceleration of B, in terms of m

(ii) the value of m if the tension in the string is 3·92 N.



(b) A moveable pulley of mass m is suspended on a light inextensible string between two fixed pulleys as shown in the diagram. Masses of 6 kg and 3 kg are attached to the ends of the string.
The system is released from rest.

(i) Show, on separate diagrams, the forces acting on the moveable pulley **and** on each of the masses.

(ii) Find in terms of m the tension in the string.

(iii) For what value of m will the acceleration of the moveable pulley be zero?

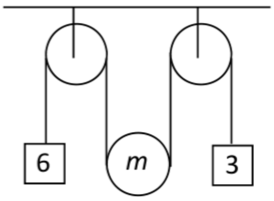


Figure 2.33

The exam question below relies more heavily on Linear Motion, and in the old syllabus was that year's Linear Motion question.

Question 2.55 — 2019 Q1(a).

1. (a) A particle P, of mass 3 kg, is projected along a rough inclined plane from the point A with speed 4·2 m s^{-1}. The particle comes to instantaneous rest at B.
The plane is inclined at an angle α to the horizontal where $\tan\alpha = \frac{9}{40}$.
The coefficient of friction between the particle and the plane is $\frac{3}{20}$.

(i) Show that the deceleration of P is $\frac{15g}{41}$.

(ii) Find $|AB|$.

After reaching B the particle slides back down the plane.

(iii) Find the speed of P as it passes through A on its way back down the plane.

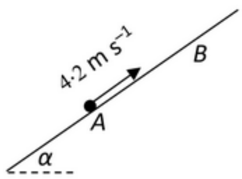


Figure 2.34

2.8 Summary

- Understand Newton's Three Laws.
- Know the various common sources of forces, i.e. gravity, normal reactions, tensions, friction, resistance forces, internal forces (e.g. an engine).
- Resistance and internal forces are uncommon in connected particles question. Instead, don't

F(o)RG(e)T your four forces common to these questions (see Note 2.20).

- Be able to follow the three steps outlined in Rule 2.21 by first drawing all your forces, creating your simultaneous equations with $F = ma$ and solving the system of equations for the desired variables.
- Understand that friction acts as a force in the opposite direction to motion, not necessarily acceleration (see Question 2.55, and that only enough frictional force (up to the limiting friction) is applied to resist motion.
- Be able to solve systems of linear simultaneous equations, and understand which variables are unknown (usually forces and acceleration) and which are not (masses and g, although there are exceptions to masses being unknown, see Question 2.54).
- The questions from Section 2.3 onwards should be considered case studies. All exam questions are some combination of the systems studied in this chapter.
- Understand that in problems where friction is not a concern, drawing a common acceleration in one direction and solving to get it as a negative number means that the objects are travelling in the opposite direction with that magnitude acceleration.
- Be aware that as one purpose of the techniques learned in this chapter is to find the acceleration of an object, these questions occasionally contain basic Linear Motion problems.

2.9 Notes on the Exam, and Work Still to Cover

On the old syllabus exam (up to 2022), Connected Particles was Question 4 of the paper.

After the syllabus change we no longer study systems where with an object on the inclined part of a wedge where the wedge can also move. Therefore if looking at exam questions from the old syllabus students should avoid these questions. They can be spotted easily; any question with a movable wedge will also give the wedge a mass.

We will also not study systems with movable pulleys that also hold a string with an object on either side (e.g. 2011 Q4 (b)), such that when the system is released from rest the objects on either side of the pulley move as the pulley itself moves. Both of these problems required knowledge of **relative acceleration** which we no longer cover.

However, overall this topic has changed little under the new syllabus and therefore old syllabus exam questions are still a great resource with which which to prepare.

We have covered all aspects of this topic.

2.10 Homework

Sources of Forces and Basic Problems

1. A man on a bicycle, with a combined mass of 120 kg, cycles with a driving force of 50 N. There is a wind resistance of 30 N. What is the acceleration of the cyclist, and how long does it take to travel 100 m if he starts from rest?

2. A boat is powered by an engine providing a driving force of 400 N, and there is a wind resistance of 100 N. Starting at rest, after 10 seconds the boat is travelling at 5 m/s. What is the mass of the boat?

3. A bullet of mass 20 grams is travelling horizontally at 200 m/s when it penetrates a fixed block of wood. It travels 1 m into the block of wood.

   (a) Find the resistance of the wood, assuming it is of uniform density. Ignore gravity.

   (b) Assume now that the block of wood is only 50 cm thick. What is the velocity of the bullet as it leaves the block of wood?

4. A ball of mass 20 kg is dropped from a height of 10 m above a patch of soft ground. It sinks 1 m into the ground before coming to rest. What is the resistance of the earth?

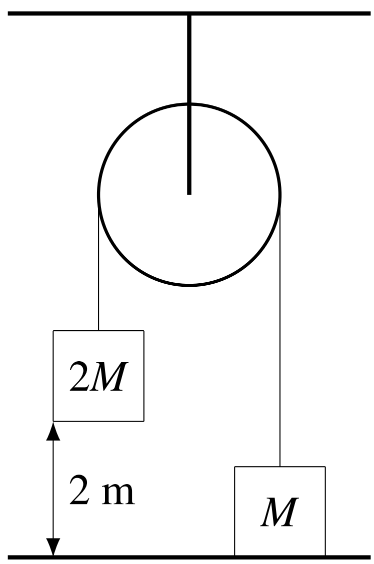
5. A particle of mass 10 kg is placed on flat ground, with a rope attached. A man pulls on the rope horizontally with a force of 100 N. The coefficient of friction between the particle and the ground is $\mu = 0.2$.

   (a) Draw a diagram showing all forces acting on the particle.

   (b) Find the acceleration of the particle.

More Advanced Problems 1

6. Two particles, of mass $2M$ and M, are connected by a taut inelastic string resting on a smooth pulley above a table as shown below. The $2M$ particle is 2 m above the table.



(a) Draw a diagram showing all the forces acting on each particle.

(b) What is the common acceleration of the particles?

(c) How long does it take for the $2M$ particle to hit the table?

(d) How much higher does the M particle rise after the $2M$ particle hits the table?

7. A particle of mass $2M$ lies on a horizontal table. It is attached by a taut string to a second particle of mass M also lying the same horizontal table. A second taut string is attached at one end to the M particle, passes over a smooth, light pulley and is attached to another particle of mass $4M$ hanging vertically, as shown in the diagram below.

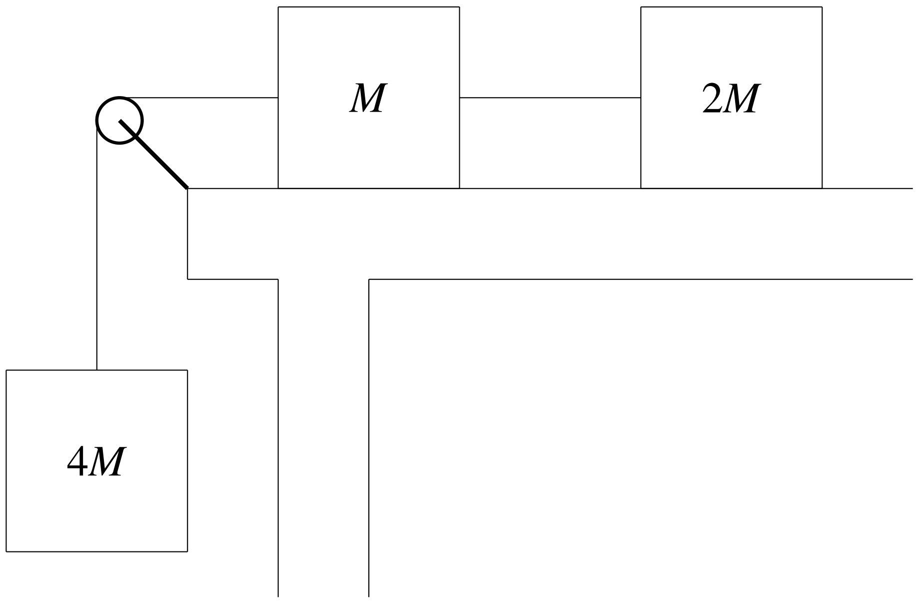


Figure 2.35

First assume that the table is smooth.

(a) Draw a diagram showing all the forces acting on each particle.

(b) Find their common acceleration.

(c) Repeat (a) and (b) when the coefficient of friction between each particle and the table is $\mu = 0.5$.

(d) Would the system act differently if instead of an M and $2M$ particle on the horizontal surface, we instead had a single $3M$ particle?

8. A particle of mass $4M$ hangs freely as it is attached to a taut string. The string passes over a light, smooth pulley and is attached at its other end by a particle of mass $2M$ which lies on a flat, horizontal table. Another string is attached to the other side of the $2M$ particle. It passes over another light, smooth pulley and is attached at its other end to a particle of mass $3M$ which hangs freely, as in the following diagram.

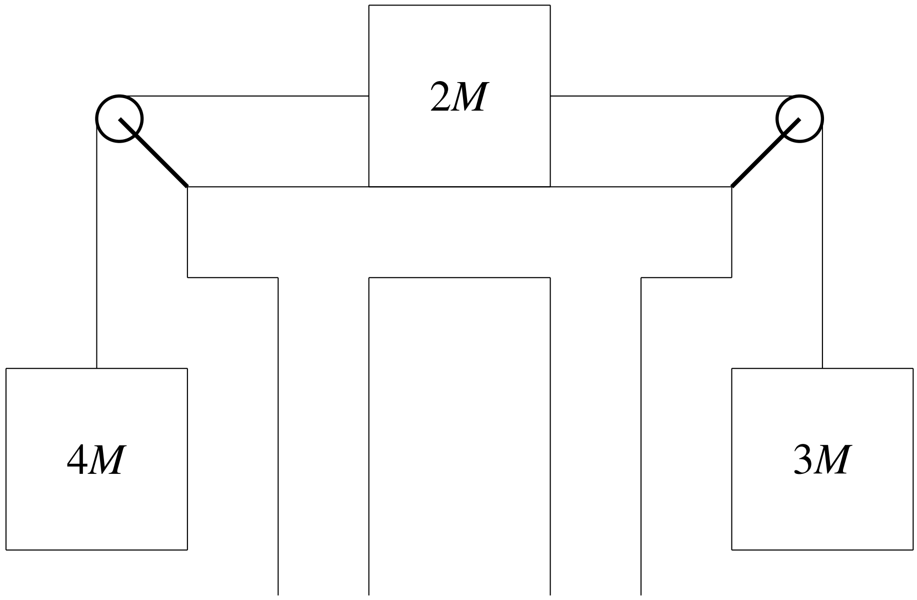


Figure 2.36

(a) Assume the table is smooth. What is the common acceleration of the particles?

(b) What is the smallest value of μ, the coefficient of friction between the $2M$ particle and the table, such that the particles do not move?

Resolving Forces

9. A block of stone with a mass of 120 kg is resting on horizontal ground. The coefficient of friction between the stone and the ground is $\mu = \frac{1}{3}$. A force of 1,000 N is applied to the block acting at an angle of 30° to the ground. Find the acceleration of the block.

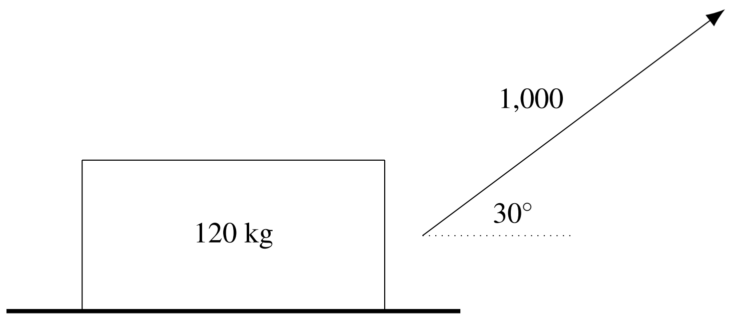


Figure 2.37

10. A block of stone with a mass of 500 kg is resting on horizontal ground. The coefficient of friction between the stone and the ground is $\mu = \frac{1}{4}$. A force of 10,000 N is applied to the block acting at an angle of α **towards the ground**, where $\tan\alpha = \frac{1}{2}$. Find the acceleration of the block.

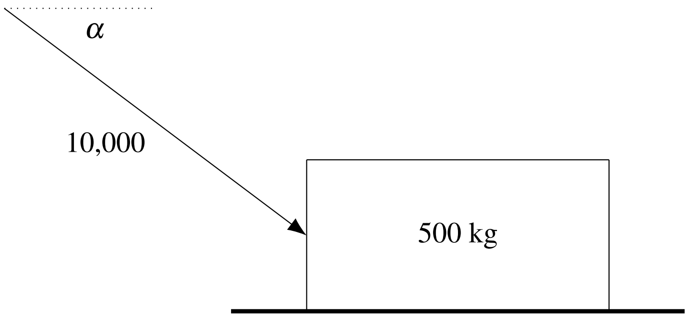


Figure 2.38

More Advanced Problems 2

11. A block of mass M lies on an immovable plane that is angled at 60° to the horizontal.

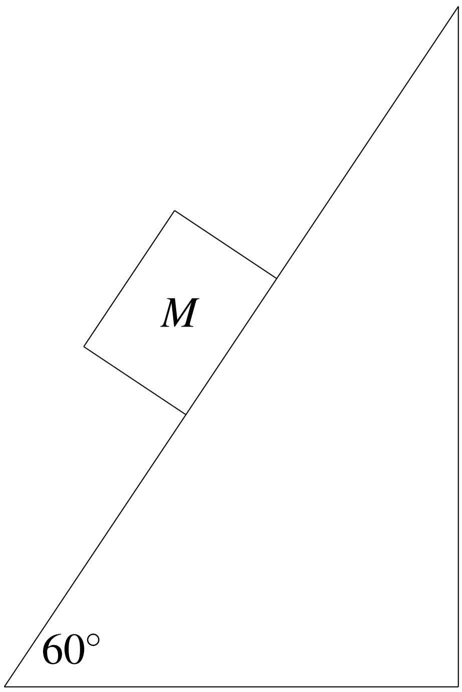


Figure 2.39

What is the acceleration of the block if

(a) The plane is smooth?

(b) The coefficient of friction between the plane and the block is $\mu = 0.5$?

(c) What is the smallest value of μ such that the block does not not move?

12. A particle of mass M lies on an immovable wedge that is angled at $\alpha = \tan^{-1}\left(\frac{3}{4}\right)$ to the horizontal. Attached to it is a light inelastic string, which passes over a pulley and is attached to an object of mass $3M$, hanging off the vertical side of the wedge without touching it. The coefficient of friction between the angled part of the plane and the block is $\mu = \frac{1}{5}$. What is the common acceleration of the particles?

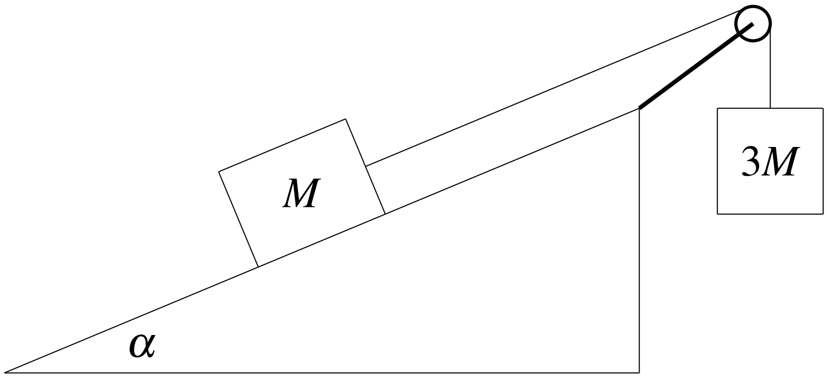


Figure 2.40

More Advanced Problems 3

13. A smooth immovable wedge has two smooth particles, one on each of its angled sides. They are connected by a light inelastic string that passes over a pulley, as shown in the diagram below.

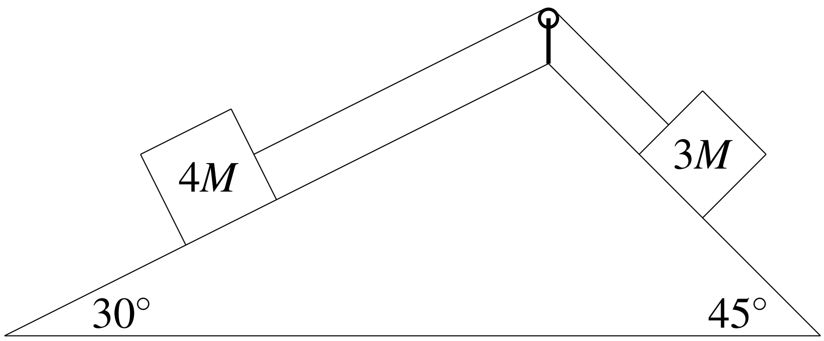


Figure 2.41

Find the common acceleration of the particles.

14. A taut string, attached at one end to a particle of mass $4M$, passes over a fixed pulley, under a moveable pulley of mass $3M$, and is attached to the ceiling at the other end. Find the acceleration of each object, making sure to clearly give direction of acceleration.

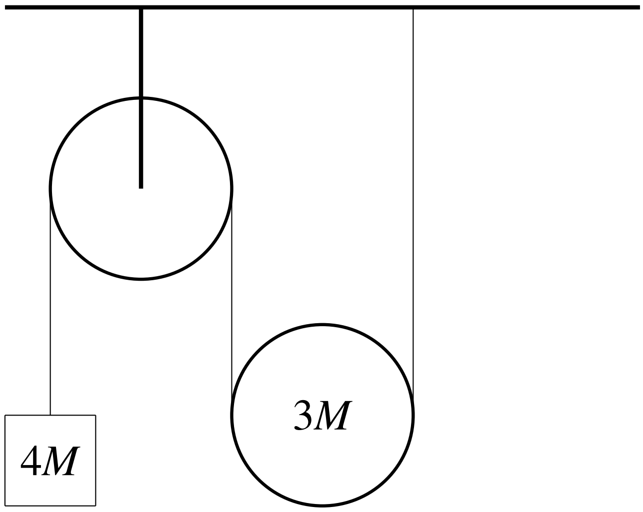


Figure 2.42

15. A string is attached on one end to a particle A of mass M. It passes over a smooth light fixed pulley, then under a smooth movable pulley B of mass $3M$, then over another light fixed pulley. It is attached at the other end to a particle C of mass $6M$. The system is released from rest.

    (a) What is the acceleration of each particle A, B and C? Make sure to give the direction of each acceleration.

    (b) What is the tension in the string?

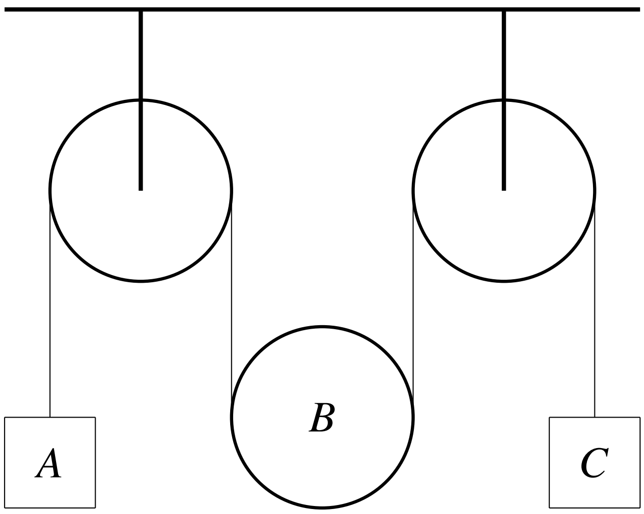


Figure 2.43

2.11 Homework Solutions

Sources of Forces and Basic Problems

1.

$$\begin{aligned}\text{Resultant Force} &= 50-30\\ &= 20\text{ N.}\\ F &= ma\\ \Rightarrow 20 &= 120a\\ \Rightarrow \frac{1}{6}\text{ m/s}^2 &= a.\end{aligned}$$

$$\begin{aligned}u &= 0\\ v &= \\ a &= \frac{1}{6}\\ s &= 100\\ t &= \end{aligned}$$

$$\begin{aligned}s &= ut+\frac{1}{2}at^2\\ \Rightarrow 100 &= \frac{1}{12}t^2\\ \Rightarrow 1200 &= t^2\\ \Rightarrow \sqrt{1200}\text{ seconds} &= t.\end{aligned}$$

2. Looking at the Linear Motion problem first,

$$\begin{aligned}u &= 0\\ v &= 5\\ a &= \\ s &= \\ t &= 10\end{aligned}$$

$$\begin{aligned}v &= u+at\\ \Rightarrow 5 &= 0+10a\\ \Rightarrow \frac{1}{2} &= a.\end{aligned}$$

$$\begin{aligned}\text{Resultant Force} &= 400-100\\ &= 300\text{ N.}\\ F &= ma\\ \Rightarrow 300 &= \frac{m}{2}\\ \Rightarrow 600\text{ kg} &= m.\end{aligned}$$

3. Looking at the Linear Motion problem first,

(a)

$$\begin{aligned} u &= 200 \\ v &= 0 \\ a &= \\ s &= 1 \\ t &= \end{aligned}$$

$$\begin{aligned} v^2 &= u^2 + 2as \\ \Rightarrow 0^2 &= 200^2 + 2a(1) \\ \Rightarrow -2a &= 40,000 \\ \Rightarrow a &= -20,000. \end{aligned}$$

$$\begin{aligned} F &= ma \\ \Rightarrow F &= 0.02(-20000) \\ &= -400 \\ \Rightarrow \text{Resistance} &= 400 \text{ N}. \end{aligned}$$

(b) Resistance and therefore deceleration is the same. So considering the path of the bullet through the block as a Linear Motion problem,

$$\begin{aligned} u &= 200 \\ v &= \\ a &= -20000 \\ s &= 0.5 \\ t &= \end{aligned}$$

$$\begin{aligned} v^2 &= u^2 + 2as \\ \Rightarrow v^2 &= 200^2 + 2(-20000)(0.5) \\ &= 20,000 \\ \Rightarrow v &= 100\sqrt{2} \text{ m/s}. \end{aligned}$$

4. Treat as multi-part journey, first part from height to ground, second part underground, with downwards direction positive.

First Part	Second Part	Extra Equations
$u_1 = 0$	$u_2 =$	$v_1 = u_2$
$v_1 =$	$v_2 = 0$	
$a_1 = g$	$a_2 =$	
$s_1 = 10$	$s_2 = 1$	
$t_1 =$	$t_2 =$	

$$
\begin{aligned}
v_1^2 &= u_1^2 + 2a_1s_1 \\
\Rightarrow v_1^2 &= 0^2 + 2(g)(10) \\
&= 20g \\
\Rightarrow v_1 &= \sqrt{20g} \\
&= 14. \\
v_2^2 &= u_2^2 + 2a_2s_2 \\
\Rightarrow 0^2 &= 14^2 + 2a_2(1) \\
\Rightarrow -2a_2 &= 196 \\
\Rightarrow a_2 &= -98. \\
F &= ma \\
\Rightarrow F &= 20(-98) \\
&= -1960 \\
\Rightarrow \text{ Resistance} &= 1,960 \text{ N}.
\end{aligned}
$$

5. (a)

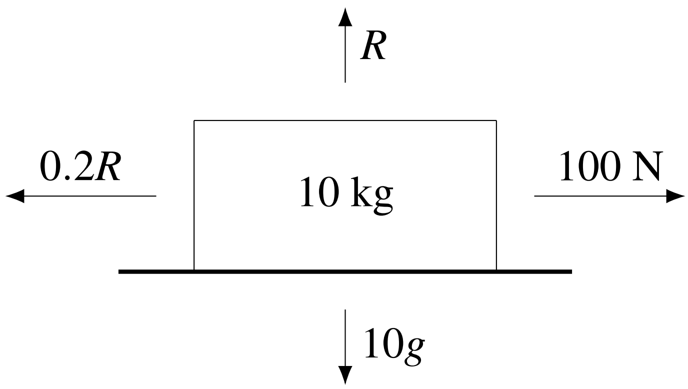


Figure 2.44

(b)

$$
\begin{aligned}
R &= 10g \\
100 - 0.2R &= 10a \\
\Rightarrow 100 - 2g &= 10a \\
\Rightarrow 8.04 \text{ m/s}^2 &= a.
\end{aligned}
$$

More Advanced Problems 1

6. (a)

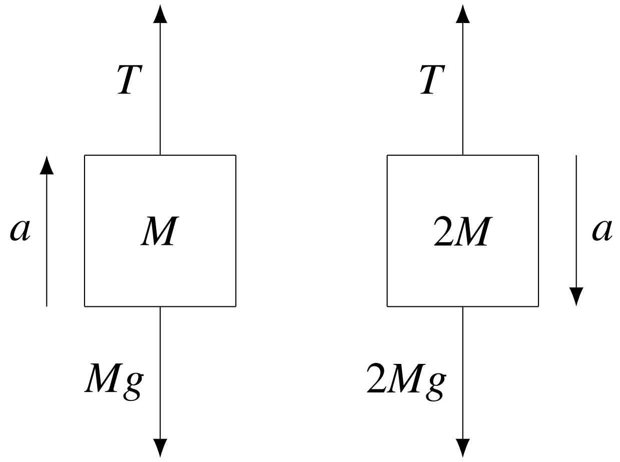


Figure 2.45

(b) $F = ma$ applied to both particles gives

$$2Mg - T = 2Ma,$$
$$T - Mg = Ma.$$

Adding these equations gives

$$Mg = 3Ma$$
$$\Rightarrow \frac{g}{3}\ \text{m/s}^2 = a.$$

(c) Considering the journey of the 2M particle from its starting position to the table as a linear motion problem,

$$u = 0$$
$$v =$$
$$a = \frac{g}{3}$$
$$s = 2$$
$$t =$$

$$s = ut + \frac{1}{2}at^2$$
$$\Rightarrow 2 = \frac{g}{6}t^2$$
$$\Rightarrow \frac{12}{g} = t^2$$
$$\Rightarrow \sqrt{\frac{12}{g}}\ \text{seconds} = t.$$

(d) In previous UVAST array,

$$\begin{aligned} v &= u + at \\ \Rightarrow v &= \frac{g}{3}\sqrt{\frac{12}{g}} \\ &= \sqrt{\frac{4g}{3}}. \end{aligned}$$

This is not just the speed of the $2M$ particle when it hits the table, it is the speed of the M particle when the string goes slack and it starts its freefall. Considering the M particles journey from when the string goes slack to when it reaches its greatest height,

$$\begin{aligned} u &= \sqrt{\frac{4g}{3}} \\ v &= 0 \\ a &= -g \\ s &= \\ t &= \end{aligned}$$

$$\begin{aligned} v^2 &= u^2 + 2as \\ \Rightarrow 0 &= \frac{4g}{3} - 2gs \\ \Rightarrow 2gs &= \frac{4g}{3} \\ \Rightarrow s &= \frac{2}{3} \text{ m}. \end{aligned}$$

7. (a)

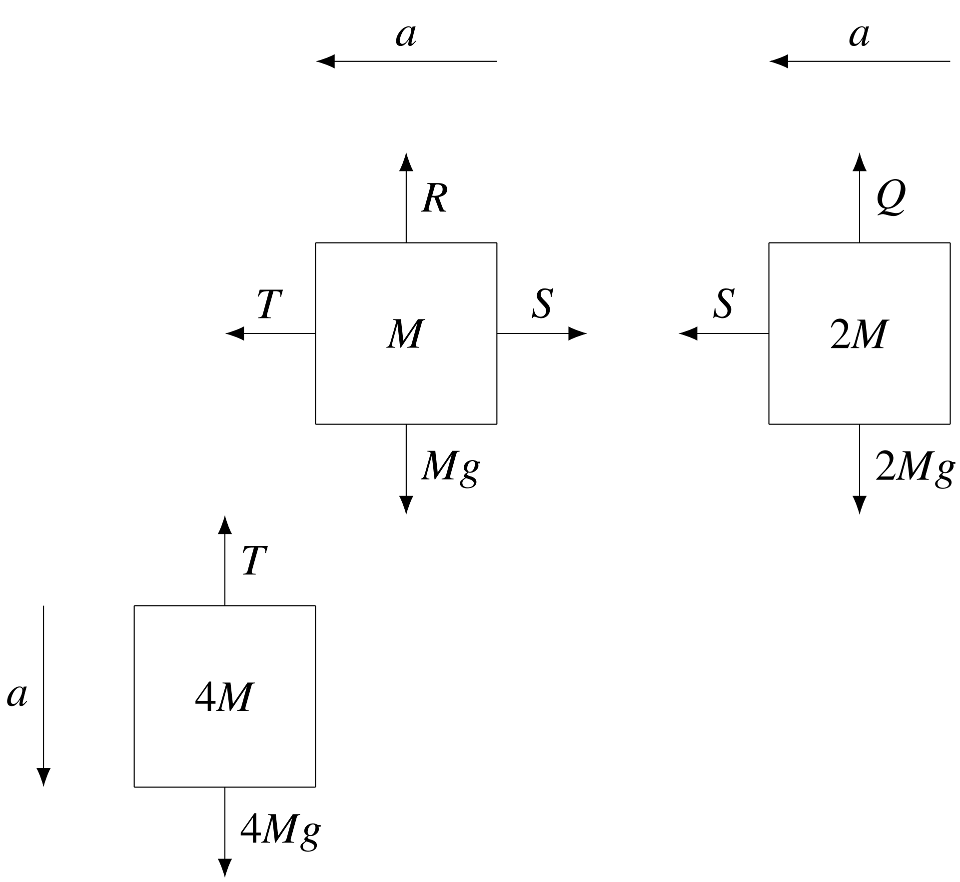


Figure 2.46

(b) Using $F = ma$,

$$\begin{aligned} 4Mg - T &= 4Ma, \\ R &= Mg, \\ T - S &= Ma, \\ Q &= 2Mg, \\ S &= 2Ma. \end{aligned}$$

Adding first, third and fifth equation,

$$\begin{aligned} 4Mg &= 7Ma \\ \Rightarrow \frac{4g}{7}\ \text{m/s}^2 &= a. \end{aligned}$$

(c)

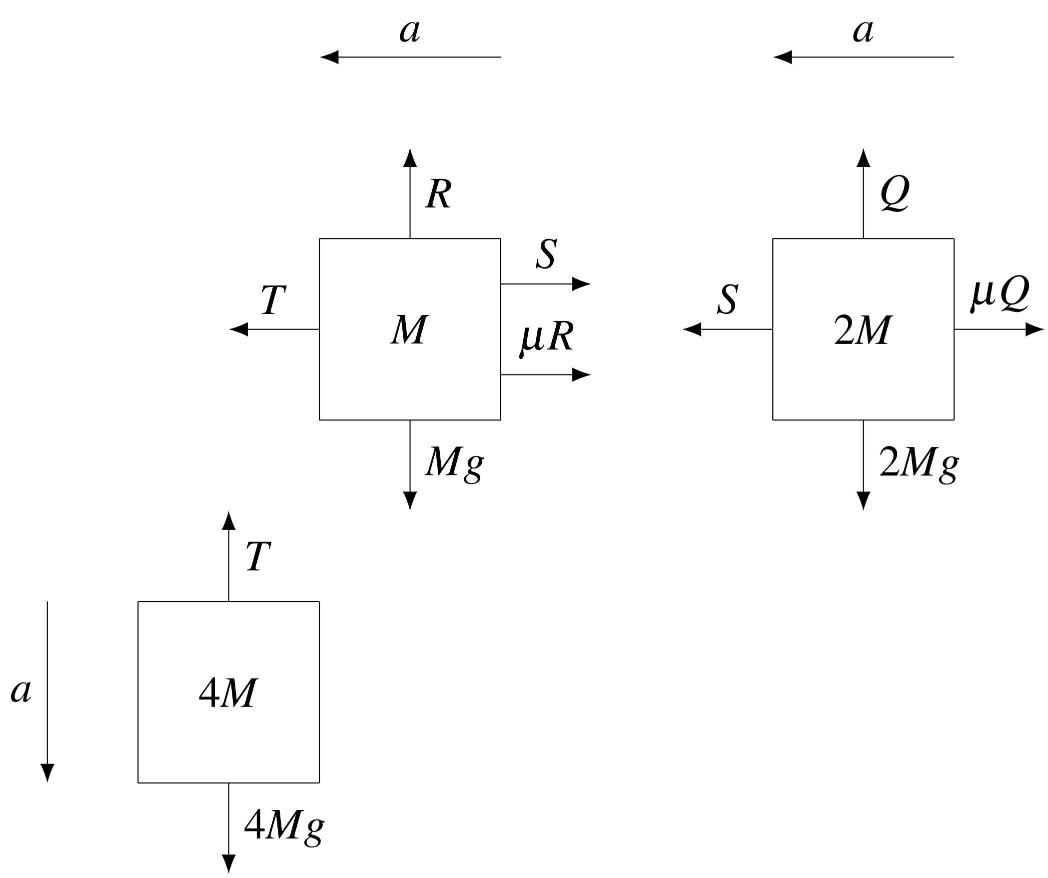


Figure 2.47

$$
\begin{aligned}
4Mg - T &= 4Ma, \\
R &= Mg, \\
T - S - 0.5R &= Ma, \\
Q &= 2Mg, \\
S - 0.5Q &= 2Ma.
\end{aligned}
$$

Substituting R and Q into equations three and five, then adding equations one, three and five,

$$
\begin{aligned}
4Mg - T &= 4Ma, \\
T - S - 0.5Mg &= Ma, \\
S - Mg &= 2Ma, \\
\Rightarrow 2.5Mg &= 7Ma \\
\Rightarrow \frac{5g}{14}\ \text{m/s}^2 &= a.
\end{aligned}
$$

(d) No. The string and the equal coefficient of friction between the M and $2M$ particle make them effectively act as one mass.

8. (a)

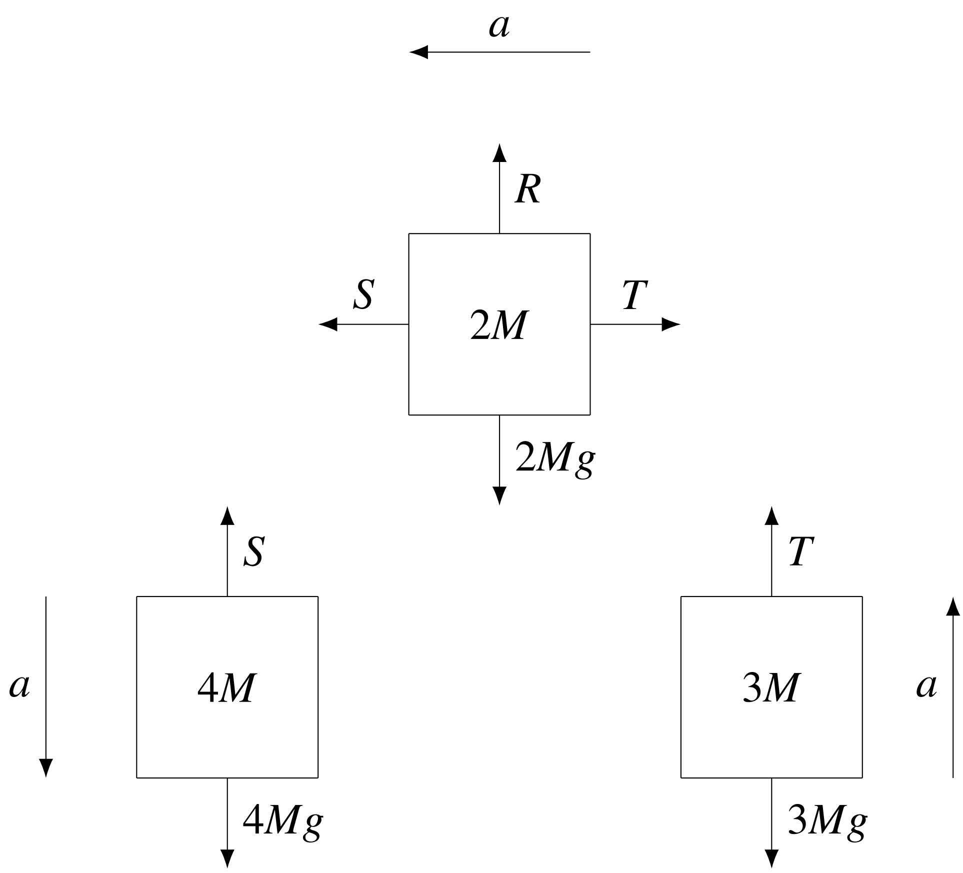


Figure 2.48

$F = ma$ gives

$$\begin{aligned} 4Mg - S &= 4Ma, \\ R &= 2Mg, \\ S - T &= 2Ma, \\ T - 3Mg &= 3Ma. \end{aligned}$$

Adding equations one, three and four,

$$\begin{aligned} Mg &= 9Ma \\ \Rightarrow \frac{g}{9}\ \text{m/s}^2 &= a. \end{aligned}$$

(b) Assume that μ is just large enough so that the frictional force is equal to the limiting friction but there is no movement of the particles. The following are the forces acting on each particle.

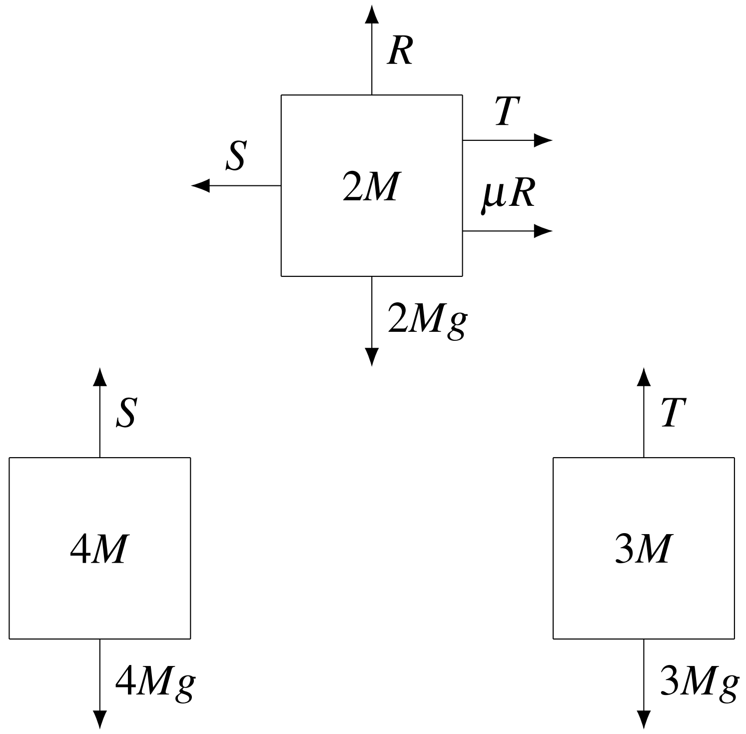


Figure 2.49

As there is no movement,

$$\begin{aligned} 4Mg &= S, \\ R &= 2Mg, \\ S &= T + \mu R, \\ T &= 3Mg. \end{aligned}$$

Substituting S, R and T into equation three,

$$\begin{aligned} 4Mg - 3Mg - \mu(2Mg) &= 0 \\ \Rightarrow Mg &= 2\mu Mg \\ \Rightarrow \frac{1}{2} &= \mu. \end{aligned}$$

Resolving Forces

9.

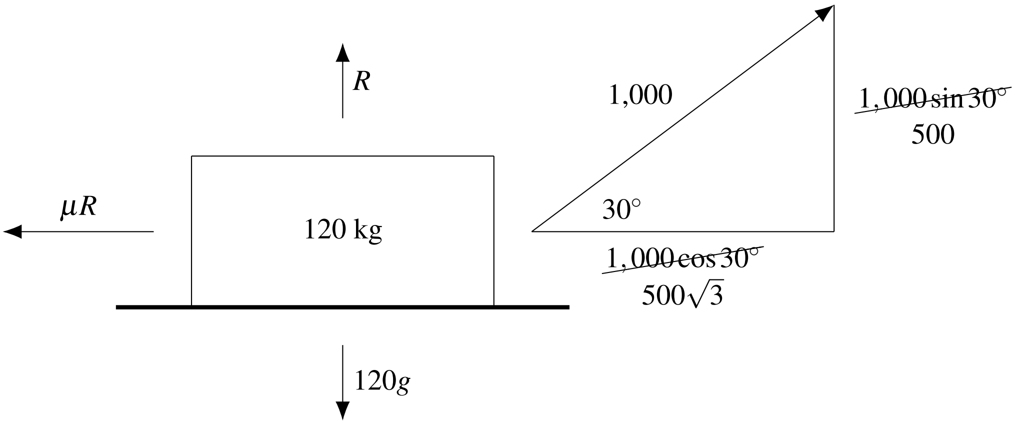


Figure 2.50

$$R + 500 = 120g$$
$$\Rightarrow R = 1176 - 500$$
$$= 676 \text{ N}.$$

If the block moves to the right with acceleration a,

$$F = ma$$
$$\Rightarrow 500\sqrt{3} - \mu R = 120a$$
$$\Rightarrow 500\sqrt{3} - \frac{1}{3}(676) = 120a$$
$$\Rightarrow 5.34 \text{ m/s}^2 = a.$$

10. From drawing a triangle it can be shown that $\cos\alpha = \frac{2}{\sqrt{5}}$, $\sin\alpha = \frac{1}{\sqrt{5}}$.

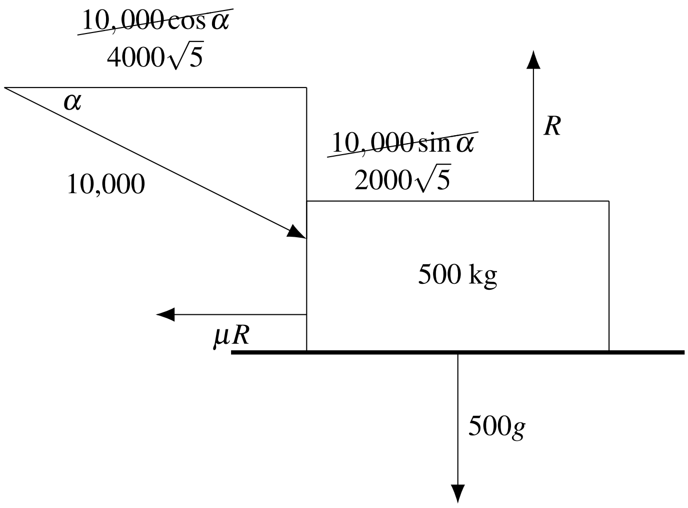


Figure 2.51

$$R = 500g + 2000\sqrt{5} \text{ N}.$$

If the block moves to the right with acceleration a,

$$\begin{aligned} F &= ma \\ \Rightarrow 4000\sqrt{5} - \mu R &= 500a \\ \Rightarrow 4000\sqrt{5} - \frac{1}{4}(500g + 2000\sqrt{5}) &= 500a \\ \Rightarrow 13.2 \text{ m/s}^2 &= a. \end{aligned}$$

More Advanced Problems 2

11. (a)

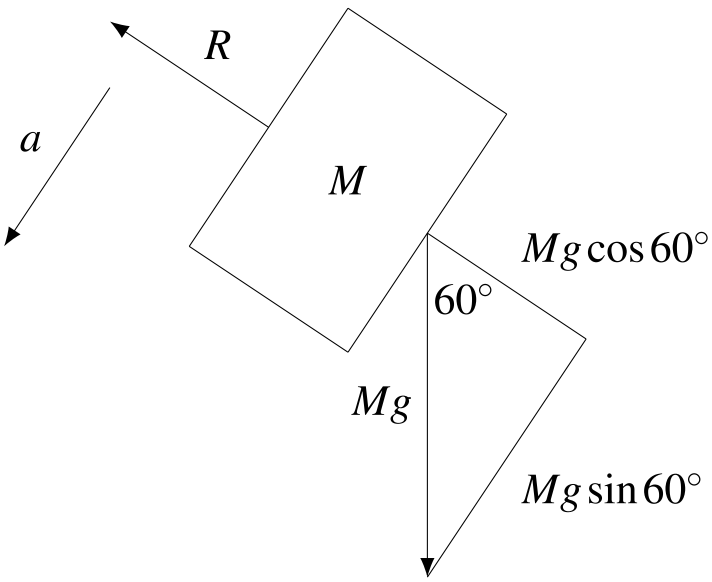


Figure 2.52

$$\begin{aligned} Mg\sin 60^\circ &= Ma \\ \Rightarrow \frac{\sqrt{3}g}{2} \text{ m/s}^2 &= a. \end{aligned}$$

(b)

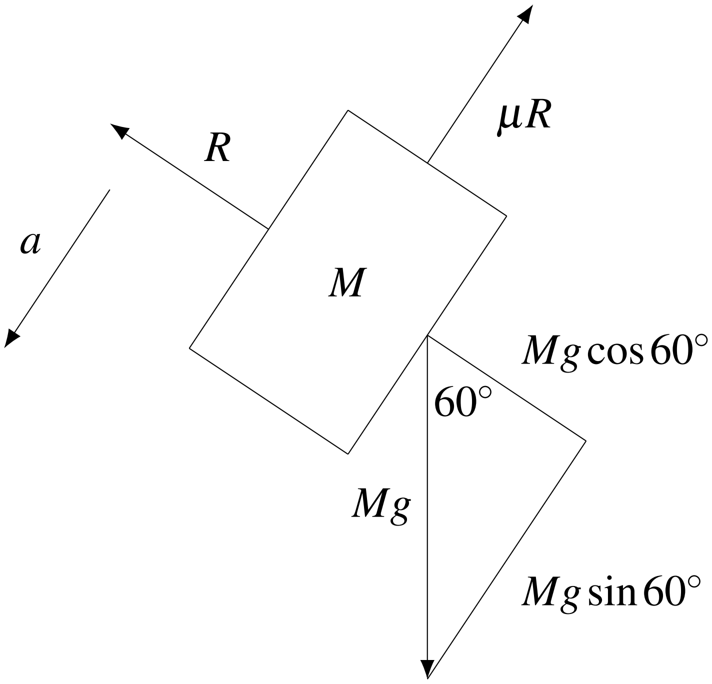


Figure 2.53

$$
\begin{aligned}
Mg\cos 60^\circ &= R, \\
Mg\sin 60^\circ - \mu R &= Ma \\
\Rightarrow Mg\sin 60^\circ - 0.5Mg\cos 60^\circ &= Ma \\
\Rightarrow \frac{2\sqrt{3}-1}{4} g \text{ m/s}^2 &= a.
\end{aligned}
$$

(c)

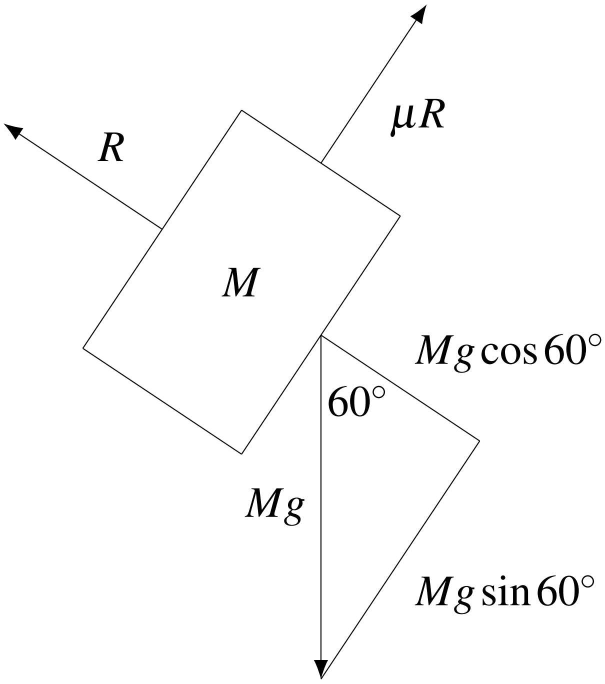


Figure 2.54

$$\begin{aligned} Mg\cos 60^\circ &= R, \\ Mg\sin 60^\circ &= \mu R \\ \Rightarrow Mg\sin 60^\circ &= \mu Mg\cos 60^\circ \\ \Rightarrow \sqrt{3} &= \mu. \end{aligned}$$

12. We can quickly calculate that $\cos\alpha = \frac{4}{5}$, $\sin\alpha = \frac{3}{5}$.

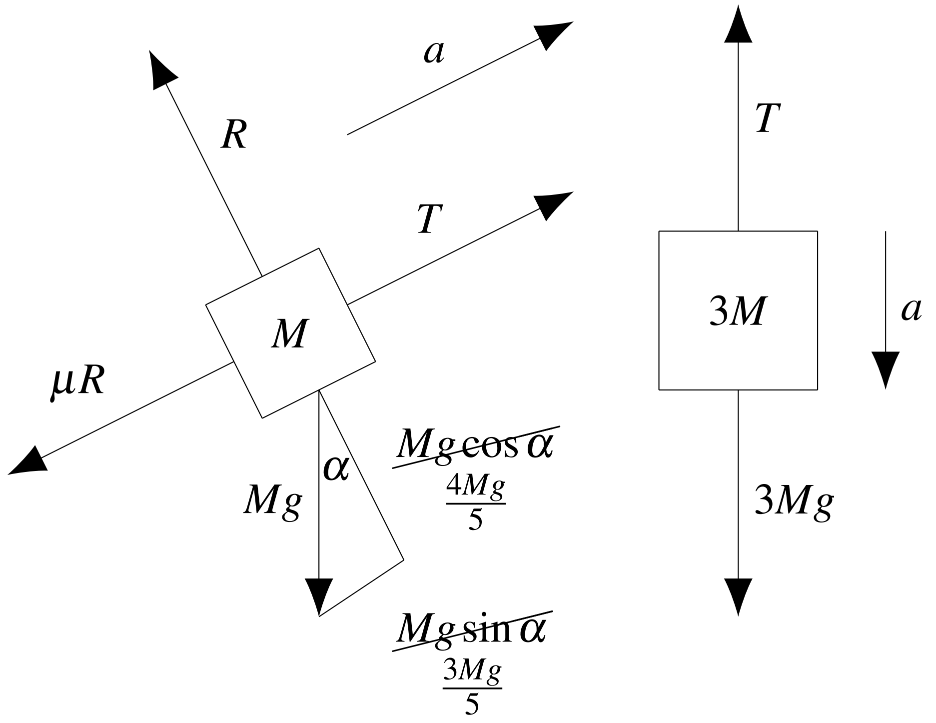


Figure 2.55

Our equations are

$$R = \frac{4Mg}{5}$$
$$T - \frac{1}{5}R - \frac{3Mg}{5} = Ma,$$
$$3Mg - T = 3Ma.$$

Replacing R in the second equation and simplifying gives

$$T - \frac{1}{5}\frac{4Mg}{5} - \frac{3Mg}{5} = Ma$$
$$\Rightarrow T - \frac{19Mg}{25} = Ma.$$

Adding the two remaining equations

$$T - \frac{19Mg}{25} = Ma,$$
$$3Mg - T = 3Ma$$

gives

$$\frac{56Mg}{25} = 4Ma$$
$$\Rightarrow \frac{14g}{25}\ \text{m/s}^2 = a.$$

More Advanced Problems 3

13. Assuming that acceleration is happening in the clockwise direction, the following show the forces and accelerations of each particle.

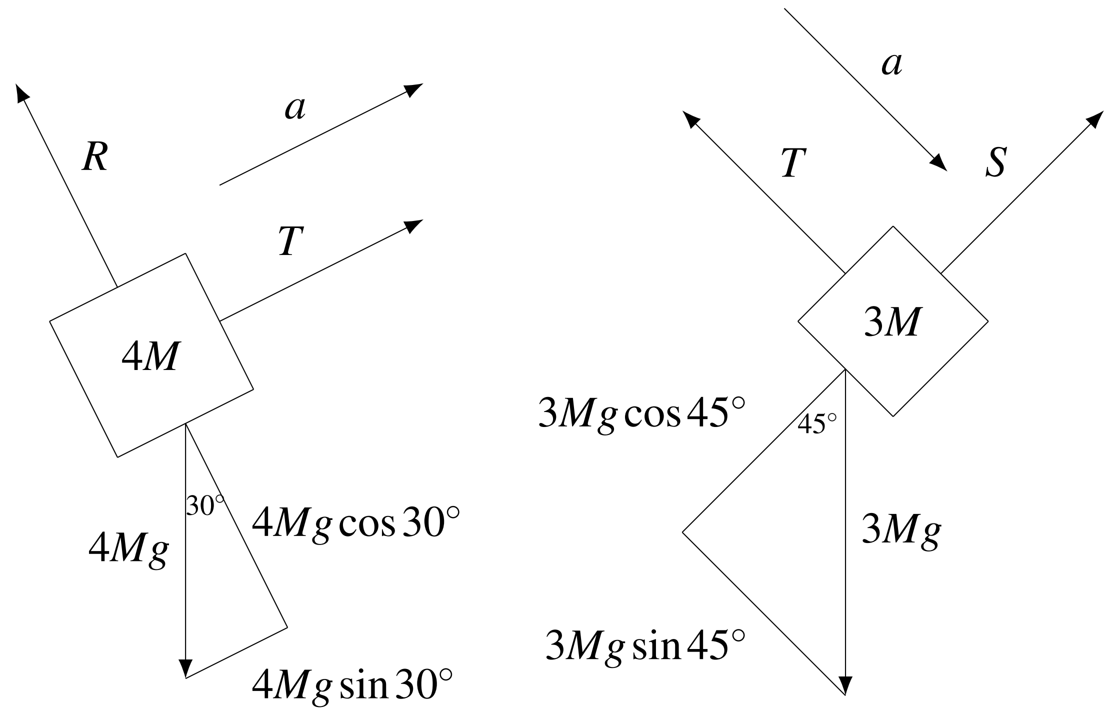


Figure 2.56

Our equations are then

$$
\begin{aligned}
R &= 4Mg\cos 30^\circ, \\
T - 4Mg\sin 30^\circ &= 4Ma, \\
S &= 3Mg\cos 45^\circ \\
3Mg\sin 45^\circ - T &= 3Ma,
\end{aligned}
$$

Ignoring the equations that give us R and S, simplifying the other two to

$$
\begin{aligned}
T - 2Mg &= 4Ma, \\
\frac{3Mg}{\sqrt{2}} - T &= 3Ma
\end{aligned}
$$

and adding them we get

$$
\begin{aligned}
\frac{3Mg}{\sqrt{2}} - 2Mg &= 7Ma \\
\Rightarrow \frac{3-2\sqrt{2}}{\sqrt{2}}Mg &= 7Ma \\
\Rightarrow \frac{3-2\sqrt{2}}{7\sqrt{2}}g \text{ m/s}^2 &= a.
\end{aligned}
$$

14.

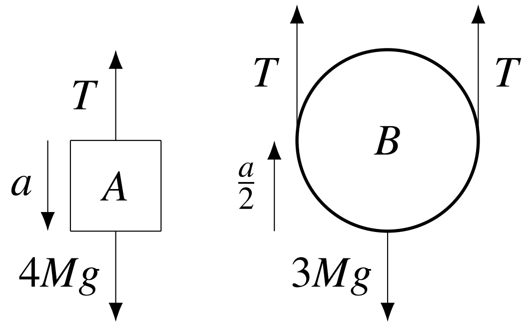


Figure 2.57

$$4Mg - T = 4Ma$$
$$2T - 3Mg = 3M\left(\frac{a}{2}\right).$$

Rewriting the first equation as

$$T = 4Mg - 4Ma$$

and inserting it into the second equation gives

$$8Mg - 8Ma - 3Mg = \frac{3Ma}{2}$$
$$\Rightarrow 5Mg = \frac{19Ma}{2}$$
$$\Rightarrow \frac{10g}{19} \text{ m/s}^2 = a.$$

15. (a)

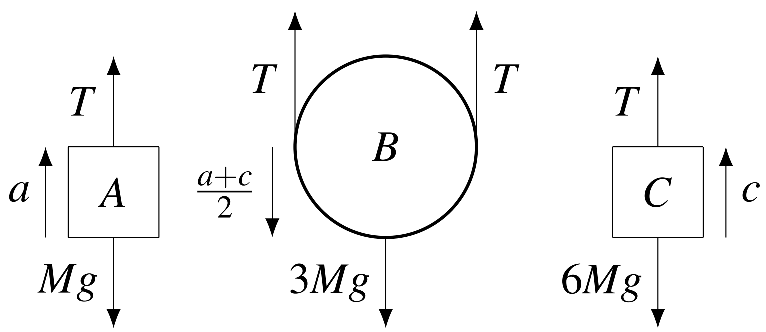


Figure 2.58

$$T - Mg = Ma$$
$$3Mg - 2T = 3M\left(\frac{a+c}{2}\right)$$
$$T - 6Mg = 6Mc.$$

Using the first equation to write T in terms of a and substituting it into the second and third equations to remove it from the system we get

$$\begin{aligned} T &= Mg + Ma \\ \Rightarrow 3Mg - 2(Mg + Ma) &= \frac{3Ma + 3Mc}{2} \\ \Rightarrow 3Mg - 2Mg - 2Ma &= \frac{3Ma + 3Mc}{2} \\ \Rightarrow 2Mg - 4Ma &= 3Ma + 3Mc \\ \Rightarrow 2Mg &= 7Ma + 3Mc \\ \Rightarrow 2g &= 7a + 3c, \end{aligned}$$

and

$$\begin{aligned} Mg + Ma - 6Mg &= 6Mc \\ \Rightarrow -5Mg &= -Ma + 6Mc \\ \Rightarrow -5g &= -a + 6c. \end{aligned}$$

Solving the system of equations

$$\begin{aligned} 2g &= 7a + 3c \\ -5g &= -a + 6c \end{aligned}$$

gives us

$$\begin{aligned} a &= \frac{3g}{5} \text{ m/s}^2 \\ c &= -\frac{11g}{15} \text{ m/s}^2. \end{aligned}$$

Therefore particle A moves upwards with acceleration $\frac{3g}{5}$ m/s^2, and particle C moves **downwards** with acceleration $\frac{11g}{15}$. Particle B moves downwards with acceleration

$$\frac{\frac{3g}{5} - \frac{11g}{15}}{2} = -\frac{g}{5} \text{ m/s}^2$$

and so it actually moves upwards with acceleration $\frac{g}{15}$.

(b)

$$\begin{aligned} T &= Mg + Ma \\ &= \frac{8Mg}{5} \text{ N}. \end{aligned}$$

2.12 Revision

As Connected Particles has changed little from the old syllabus, doing revision questions written by me is unnecessary; students should instead be practicing exam questions, of which there are decades worth.

On the 2023 exam, Connected Particles appeared in Question 5 (a). On the sample paper Connected Particles appeared in Question 7 (b). Before that, Connected Particles appeared in Question 4 of the exam paper. The list below shows the Connected Particles exam questions (post 1996) that students shouldn't attempt as they are no longer on the syllabus, or because they require work from sections not yet covered.

2022 Q4 (b) (no longer on syllabus)
2021 Q4 (b) (no longer on syllabus)
2019 Q4 (b) (no longer on syllabus)
2017 Q4 (b) (no longer on syllabus)
2015 Q4 (a) (iii) (requires Collisions)
2015 Q4 (b) (no longer on syllabus)
2014 Q4 (b) (no longer on syllabus)
2011 Q4 (b) (no longer on syllabus)
2010 Q4 (b) (no longer on syllabus)
2008 Q4 (b) (no longer on syllabus)
2006 Q4 (b) (no longer on syllabus)
2004 Q4 (b) (no longer on syllabus)
2002 Q4 (a) (no longer on syllabus)
2001 Q4 (no longer on syllabus)
2000 Q4 (b) (no longer on syllabus)
1999 Q4 (b) (no longer on syllabus)
1996 Q4 (no longer on syllabus)

Solutions to these questions are available on examination.ie/exammaterialarchive/ or at brendan-williamson.ie/pastpapers.