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Contents

This book is designed to be read as a single document, in the order in which the chapters appear. In each chapter questions, examples, definitions etc. are on a common numbering system, meaning that Definition 4.7 is the 7th text box in Chapter 4. Similarly figures are on a separate common numbering system, so that Figure 7.12 is the 12th picture in Chapter 7. This book also appears in pdf format on Moodle, in which all references are hyperlinked for easier navigation.

At the end of each chapter is a summary of the major points, which are more relevant to a re-reading of the chapter than when the material is first studied. There is also a set of homework problems (separated by section) with solutions provided at the end of the chapter. This is to be completed as we work through the chapter. There is also a revision section at the end of each chapter for students re-reading the chapter or preparing for a test.

This book is comprehensive, assuming no knowledge of Maths beyond the basics of Junior Cycle and so you don't need any other book or notes to study for the course. It is also designed so that it can be read as revision weeks or months after we first cover them. As such you shouldn't need to spend much time taking notes and can instead concentrate on the class. However you may want to take notes occasionally if there is something mentioned in class that isn't covered clearly in the book.

Tests will take place shortly after we complete a chapter, and will be of a similar difficulty to homework and revision problems.

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Algebra 1

1. Brackets & Factorising

1.1 Some Terms

Before we delve into any material we will clarify some terms that we will use throughout the year.

Definition 1.1 A **term** is either a single number or variable, or multiple numbers and/or variables multiplied together. The number in front of the term is called the **coefficient**. If there is no algebraic component the term is called a constant, or constant term.

For example, 2, $3x$, $-x^2$ and xy are terms, with coefficients 2, 3, -1 and 1, but $2+x$ is not. 2 is a constant term.

Definition 1.2 An expression is a combination of one or more terms added or subtracted together.

For example, $2+x$, $3-4x+5y$, $-2z$ are expressions. A single term is still an expression.

Definition 1.3 An **equation** is two expressions combined with an equals sign.

For example,

$$\begin{aligned} 3 &= x \\ x^2-5x+6 &= 0 \\ x^2+y^2 &= z^2 \end{aligned}$$

are equations.

Definition 1.4 An **identity** is an equation that is true for every value of the variables in the equation.

For example,

$$\begin{aligned} 2 &= 2 \\ x^2 &= x \times x \end{aligned}$$

are identities; they are statements of mathematical fact.

$$x+2=3$$
$$x^2-5x+6=0$$

are equations; they are only true for certain values of x.

1.2 Adding and Multiplying Terms

As a reminder, the term $4x^2$ is "4 multiplied by x^2", or, broken down further, "4 multiplied by x multiplied by x". More mathematically,

$$4x^2 = 4 \times x \times x.$$

We will sometimes refer to x^2 as "x raised to the power of 2". When it comes to adding terms, we can only add **like** terms; i.e. terms with the same variables raised to the same powers. For example,

$$3x+4y,$$
$$5x^2+7x,$$
$$3x-4xy$$

can't be added together (or subtracted) as they are respectively different variables, the same variable raised to a different power, and a different combination of variables. On the other hand,

$$3x+4x=7x$$
$$5x^2+7x^2=12x^2$$
$$3xy-4xy=-xy$$

can be added together or subtracted as they are the same terms.

Question 1.5 For each of the following terms, if they cannot be added or subtracted explain why. If they can, add/subtract them.

$$5x-2x,$$
$$7x^2-5x^3,$$
$$3x^2y+8x^2y,$$
$$5xy^2-2x^2y,$$
$$x^3+5x^3-2x^3,$$
$$3x+4x-5y.$$

When it comes to multiplying terms, we can multiply any two terms together. To do so we multiply the same variables, and constants, together. The constants multiply as normal, and to multiply the variables we add the powers, treating x as x^1. For example,

$$4x^2 \times 2x^3 = 8x^5,$$
$$3x^2y \times 5x^4y^2 = 15x^6y^3,$$
$$-4a \times 7ab = -28a^2b,$$
$$-3p \times -4q = 12pq,$$
$$8 \times 2xy = 16xy.$$

Question 1.6 Multiply the following terms.

$$3x^2 \times 5x$$
$$4 \times 3z$$
$$xyz \times x^3 z^2$$
$$5x \times -2$$
$$4x^2 y \times 4xy^2$$

1.3 Factors of a Term

Now that we understand multiplying terms we can consider another definition.

Definition 1.7 A factor of a term is a term that can be multiplied by another term with whole number coefficients to get the original term.

For example, 6 and x^3 are factors of $6x^3$. Each "part" of a term is always a factor. However x and x^2 are also factors, as

$$x \times 6x^2 = 6x^3,$$
$$x^2 \times 6x = 6x^3.$$

2 and 3 are also factors of $6x^3$, as

$$2 \times 3x^3 = 6x^3,$$
$$3 \times 2x^3 = 6x^3.$$

$2x$ is also a factor, but $4x$ is not as

$$2x \times 3x^2 = 6x^3$$
$$4x \times 1.5x^2 = 6x^3$$

but the second term can't be a decimal or fraction.

In truth finding factors doesn't require guesswork. Every term can be broken down into its simplest components by reducing the constant to its prime factors, and each variable term into single powers. Every factor is then just a product of some or all of these factors. For example,

$$6x^3 = (3)(2)(x)(x)(x).$$

Any collection of these can be combined to make a factor. For example, $(2)(x) = 2x$ and $(3)(2)(x)(x) = 6x^2$ are factors.

For larger numbers, breaking a number down into its prime factors can be done by breaking numbers into two parts until you no longer can.

Example 1.8 Break $48x^2y^3z$ down to its simplest components and give two factors of it.

$$\begin{aligned}
48x^2y^3z &= (12)(4)(x)(x)(y)(y)(y)(z)\\
&= \widehat{(4)(3)}\widehat{(2)(2)}(x)(x)(y)(y)(y)(z)\\
&= \widehat{(2)(2)}(3)(2)(2)(x)(x)(y)(y)(y)(z)\\
&= (2)(2)(2)(2)(3)(x)(x)(y)(y)(y)(z).
\end{aligned}$$

The $\widehat{\text{wide hat}}$ is showing that the terms under the hat were one number on the previous line. The last line just rearranged the constants from smallest to largest.

There are lots of factors of $48x^2y^3z$, such as

$$\begin{aligned}
(2)(2)(3)(x)(x)(y) &= 12x^2y,\\
(2)(x) &= 2x.
\end{aligned}$$

Question 1.9 Break each of the following terms down to their simplest components.

$$\begin{array}{l} 18x^2 \\ 35x^3y^2 \\ 60ab^2c^3 \end{array}$$

For each term give two factors.

You may have noticed that there is a simple way to see if one term is a factor of another term.

Theorem 1.10 A term F is a factor of another term T if and only if the coefficient of F divides the coefficient of T, and each variable present in F is present in T to an equal or higher power.

For example, we can tell that $12x^2y$ is a factor of $48x^2y^3z$ because

- 12 divides 48 (i.e. $\frac{48}{12} = 4$ is a whole number),
- there is an x^2 in $12x^2y$ and an x^2 in $48x^2y^3z$,
- there is a y in $12x^2y$ and a y^3 in $48x^2y^3z$.

However $18x$ is not a factor of $48x^2y^3z$ as 18 does not divide 48. x^3 is also not a factor as 3 is greater than 2, and $4xz^2$ is not a factor as 2 is greater than 1 in the powers of z.

Question 1.11 Use Theorem 1.10 to check if each of the terms on the left are factors of the terms on the right.

$$\begin{array}{ll}
5x^2 & 15x^3\\
4x^3 & 12x^3\\
5x & 5x^2y\\
3x^2y & 12xy^2\\
8x & 12xy^2\\
5y & 10x^5\\
7x^2yz^3 & 28x^3yz^7
\end{array}$$

Although Theorem 1.10 is a great test to check if a term is a factor of another term, it is important to remember Definition 1.7. In particular, we can use Theorem 1.10 to show that $5x^2$ is a factor of $15x^3$, but by Definition 1.7 what exactly do we multiply $5x^2$ by to get $15x^3$? The answer is $3x$, but how do we know (other than guessing)? When we break $15x^3$ into its simplest components we get

$$\begin{aligned} 15x^3 &= (3)(5)(x)(x)(x). \\ 5x^2 &= (5)(x)(x), \\ \text{and } 3x &= (3)(x). \end{aligned}$$

Notice that $3x$ is exactly the product of the components of $15x^3$ that are leftover when we take some components to make $5x^2$. From here, given an original term like $15x^3$ and a factor like $5x^2$ we will call terms like $3x$ the leftover term.

Thankfully we have a simpler way of finding the leftover terms.

Theorem 1.12 If F is a factor of T with leftover term L (i.e. $T = F \times L$), the coefficient of L is the coefficient of T divided by the coefficient of F, and for each variable in T the power of this variable in L is equal to the power of the variable in T minus the power of the variable in F. Powers of zero are treated as absent variables and absent variables are treated as powers of zero.

Applying Theorem 1.12 to $15x^3$ and $5x^2$, see that $\frac{15}{5} = 3$. Regarding the powers of x, $3 - 2 = 1$, so the leftover term is $3x^1 = 3x$. You can remember this as **dividing** the coefficients and **subtracting** the powers.

Example 1.13 Find the leftover term when we factor out $4x^2y$ from $20x^2y^3z$.

It's $5y^2z$, as $\frac{20}{4} = 5$. Notice that for the x component, we don't include x^0.

This may be understood more intuitively as seeing what's left in $20x^2y^3z$ when we "pull out", or "factor out" $4x^2y$. If we factor a 4 out of $20 = 4 \times 5$ then only the 5 is left. If we pull out x^2, which is two x's multiplied together, then there are none left in $20x^2y^3z$. When we pull out y there are two y's, i.e. a y^2, left. We left the z alone, so it is also left.

Question 1.14 The terms on the left are factors of the terms on the right. Find the leftover term in each case.

$12x$	$24x^3$
$5x$	$20x$
4	$20x$
$5x^2y$	$5x^2y^3$
xy^2	$13x^3y^2z$

It is straightforward to check if we have found the leftover term correctly. We claimed that

$$20x^2y^3z = 4x^2y \times 5y^2z.$$

We can see from our earlier work multiplying terms that this is true.

Question 1.15 Your friend was given a question like Question 1.14 for homework and wants you to check his answers. In each case multiply the factor by the leftover term to check if he is correct.

Factor	Original Term	Leftover Term
$5x$	$20x^3$	$4x^2$
$3x^2$	$9x^3y$	$3x$
$5xy$	$10x^2y^2$	$5xy$
x^2y	$4x^3yz$	$4xz$.

1.4 Highest Common Factors of Terms

Given two terms, for example

$$12x^3 \qquad 8x^5,$$

$2x$ is a factor of both terms. So is $4x^2$. What is the "largest factor" of both terms? And what do we mean by largest?

Definition 1.16 Given two (or more) algebraic terms, the **Highest Common Factor** (or **HCF**) of the terms is the factor of all terms with the largest coefficient and highest power of each variable.

Theorem 1.17 Given two or more algebraic terms, the highest common factor can be found in the following way. First, the coefficient of the HCF is equal to the highest common factor of the coefficients of the terms. Then, for each variable appearing in all terms, the power of this variable in the HCF is its minimum power over all terms.

For example, given the two terms

$$12x^3 \qquad 8x^5,$$

the highest common factor of 12 and 8 is 4. Regarding x, the minimum of 3 and 5 is 3. Therefore the HCF is $4x^3$.

Example 1.18 Find the HCF of the following two terms.

$$15x^2y^3 \qquad 3xy^4$$

The HCF of 15 and 3 is 3. Using the lowest of the two powers for x, and then for y, the HCF is $3xy^3$.

We can apply this logic to more than two terms.

Example 1.19 Find the HCF of

$$12x^3 \qquad 3x^5 \qquad 15x$$

It's $3x$, as the HCF of 12, 3 and 15 is 3, and regarding x the minimum of 3, 5 and 1 is 1.

It is straightforward to tell if the common factor you found is actually the HCF. If you find the leftover terms after factoring out this factor, they should have a HCF of 1 (sometimes having a HCF of 1 is referred to as having "no common factors"). This means that their coefficients have no common factors and they have no variables in common. For example, considering our examples above,

Terms			HCF	Leftover Terms		
$12x^3$	$8x^5$		$4x^3$	3	$2x^2$	
$15x^2y^3$	$3xy^4$		$3xy^3$	$5x$	y	
$12x^3$	$3x^5$	$15x$	$3x$	$4x^2$	x^4	5

See in each row the coefficients of the leftover terms have no common factors (except 1) and no variables in common.

Question 1.20 For each of the following terms, find the HCF and confirm that it is the HCF by finding the leftover terms.

Terms			HCF	Leftover Terms
$5x^7$	$10x^2$			
$10x^2$	$10x^3$			
$10x^3y^2$	$15xy^6$			
$12x^3y^2z$	$15xy^2z^3$	$27x^5y^2$		
$7x^5y$	$4x^3$			

What if you don't have a HCF of 1? Say we attempt to find the HCF and leftover terms for the following terms and get the following answer.

Terms		HCF	Leftover Terms	
$12x^5y^2z$	$18x^2yz$	$3x^2y$	$4x^3yz$	$6z$

The leftover terms have a HCF of $2z$, which means we made a mistake. We combine the $2z$ with our existing HCF by multiplying to get the actual correct answer.

Terms		HCF	Leftover Terms	
$12x^5y^2z$	$18x^2yz$	$6x^2yz$	$2x^3y$	3

Question 1.21 Your friend tried to find the HCF of some terms below. See if he made any mistakes by checking if the leftover terms had any common factors, and if so find the real HCF and the real leftover terms.

Terms			HCF	Leftover Terms		
$8x^5$	$4x^3$		$2x^3$	$4x^2$	2	
$9x^3y$	$3x^2y^2$		$3x^2y$	$3x$	y	
$5x^8y^2$	$15x^6yz$		x^6	$5x^2y^2$	$15yz$	
$3x^2yz$	$9xy^2z$	$15x^3y^3z$	$3xy$	xz	$3yz$	$5x^2y^2z$

1.5 Expanding Brackets

In Junior Cycle you learned how to expand brackets. For example,

$$5(3+x) = 15+5x,$$
$$2x(2x-5y) = 4x^2-10xy.$$

There was a reason for this; it was more than just a "rule". Regarding the first equation, imagine the following rectangular field split into two parts where one of the measurements is unknown.

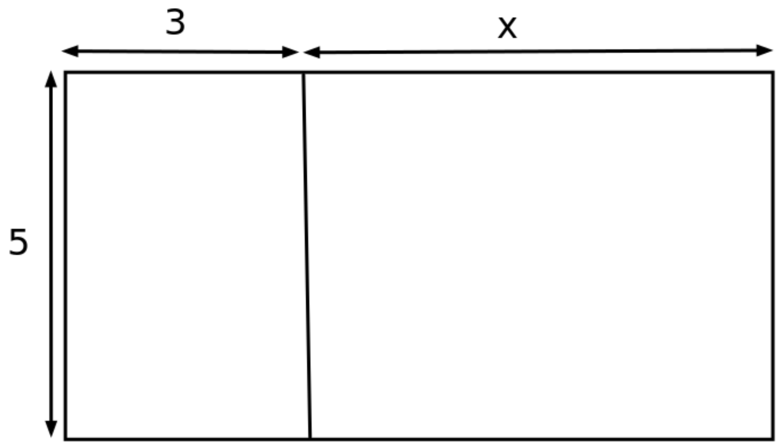


Figure 1.1

We could calculate the area of the field in one way, by considering the total width multiplied by the height, getting $5(3+x)$. Or, we could find the area of each part of the field individually and add them together, getting $15+5x$. As both methods calculate the same area, both answers are equal. In principle, we won't imagine rectangles every time we see brackets, instead we will just obey the rules

$$a(x+y) = ax+ay,$$
$$a(x+y-z) = ax+ay-az,$$
$$\text{etc.},$$

i.e. we multiply the number on the outside by each number on the inside (and keep the signs).

Question 1.22 Expand out the following brackets.

$$3(4+2x)$$
$$5(7x-5)$$
$$3x(4x^2-3x+2)$$
$$-2y(-3+x+4y)$$

1.6 Factorising Using the HCF

Given the answers in Question 1.22, how do we turn them into the original, bracketed expressions? For example, given $6x+4$, or $3x^2-5x$, how do we know that they can be written as

$$6x+4 = 2(3x+2),$$
$$3x^2-5x = x(3x-5)?$$

The answer is that we "pull out", or "factor out" the HCF of the terms. For example, the HCF of $6x$ and 4 is 2, with leftover terms $3x$ and 2. The HCF of $3x^2$ and $5x$ is x, with leftover terms $3x$ and -5 (we keep the minus sign).

Example 1.23 Factorise $10x^3y^2+15xy^6$.

$$10x^3y^2 + 15xy^6 = 5xy^2(2x^2 + 3y^4).$$

Note 1.24 Notice that Example 1.23 and Question 1.20 are asking us to perform identical calculations. In fact the terms in the third problem in Question 1.20 are the same as those in Example 1.23.

The answer to Example 1.23 can be checked on two fronts. First, see that the equality is correct as if we multiply out the brackets on the right we do get the expression on the left. Second, the leftover terms inside the bracket have no common factors. If the brackets correctly multiply out correctly but the leftover terms have a common factor, we can continue to factor out until they no longer have common factors. Consider the following example, where we factorise in stages.

Example 1.25 Factorise $24x^5 + 60x$.

Say we start by just factoring out the x and a 2 because we noticed both numbers are even.

$$24x^5 + 60x = 2x(12x^4 + 30).$$

This is correct, but the leftover terms have a common factor. Say we only notice that they have a common factor of 2 because 12 and 30 are both even. We can factor out a 2 and multiply it to the previous factor.

$$\begin{aligned} 24x^5 + 60x &= 2x(12x^4 + 30) \\ &= 2x(2)(6x^4 + 15) \\ &= 4x(6x^4 + 15). \end{aligned}$$

Now that the numbers are smaller we can notice they they are both divisible by 3, so that

$$\begin{aligned} 24x^5 + 60x &= 2x(12x^4 + 30) \\ &= 2x(2)(6x^4 + 15) \\ &= 4x(6x^4 + 15) \\ &= 4x(3)(2x^4 + 5) \\ &= 12x(2x^4 + 5). \end{aligned}$$

It is not recommendeded to rely on this method, but if you're having trouble with applying Theorem 1.17 you don't need to find the HCF "all in one go".

As a note on language, when

$$24x^5 + 60x$$

is written as

$$2x(12x^4 + 30)$$

it is considered **factored** as the expression $24x^5 + 60x$ has been written as the product of two factors. Even though $24x^5 + 60x$ is an expression and not a term we still consider $2x$ and $12x^4 + 30$ as factors as they can both be multiplied by another expression (with whole number coefficients) to get $24x^5+60x$. However it is not considered **fully factored** as the non-term expression can be factorised further.

Definition 1.26 An expression is considered **fully factored** if it is written as the product of factors where no non-term factor can be factored further.

Therefore

$$24x^5+60x = 12x(2x^4+5)$$

is considered fully factored as $2x^4+5$ cannot be factored further. We only consider non-term factors as splitting up $12x$ into $(12)(x)$ or $(3)(4x)$ is not useful.

Question 1.27 Fully factorise the following expressions.

$$\begin{aligned}
&12x^3+6x^4\\
&4x^3-4x\\
&8xy^2-6x^3z\\
&12x^3yz-9xy^4z^2+6xz\\
&18x^3z-9x^2y+8x
\end{aligned}$$

1.7 Expanding Multiple Brackets

Earlier in the chapter we learned how to multiply brackets. Because of how multiplication works, we could have the outside term on the right rather than the left, i.e.

$$2x(3x-5) = 6x^2-10x$$

but

$$(3x-5)2x = 6x^2-10x$$

as well. What is important is that every term inside the bracket is multiplied by the term outside the bracket.

Example 1.28 Fully expand $(4x+5)(3y+7)$.

Let's first consider $(4x+5)$ as "the bracket". Then we have to multiply every term inside $(4x+5)$ by $(3y+7)$. More mathematically,

$$(4x+5)(3y+7) = 4x(3y+7)+5(3y+7).$$

Continuing as we learned before,

$$\begin{aligned}
(4x+5)(3y+7) &= 4x(3y+7)+5(3y+7)\\
&= 12xy+28x+15y+35.
\end{aligned}$$

We could also consider $(3y+7)$ as the bracket:

$$\begin{aligned}
(4x+5)(3y+7) &= (4x+5)3y+(4x+5)(7)\\
&= 12xy+15y+28x+35
\end{aligned}$$

as before (albeit in a different order). These workings can be summarised as "every term inside the first bracket is multiplied by every term inside the second bracket".

To see another example involving minus signs and more than two terms, consider the following example.

Example 1.29 Fully expand $(3x-5+4y)(5z-3)$.

$$\begin{aligned}(3x-5+4y)(5z-3) &= 3x(5z-3)-5(5z-3)+4y(5z-3)\\ &= 15xz-9x-25z+15+20yz-12y\end{aligned}$$

or

$$\begin{aligned}(3x-5+4y)(5z-3) &= (3x-5+4y)5z+(3x-5+4y)(-3)\\ &= 15xz-25z+20yz-9x+15-12y.\end{aligned}$$

As another note on language, the brackets/expression are **expanded** on the first line but are not considered **fully expanded** until the brackets are gone completely.

Sometimes like terms can be added/subtracted after expressions are fully expanded.

Example 1.30 Fully expand $(3x-5)(5x+4)$. Combine like terms where possible.

$$\begin{aligned}(3x-5)(5x+4) &= 15x^2+12x-25x-20\\ &= 15x^2-13x-20.\end{aligned}$$

Question 1.31 Fully expand out the following expressions. Combine like terms where possible.

$$\begin{aligned}&(4x+2)(3+2y)\\ &(x+1)(5-y)\\ &(3x^2-z)(5y+4)\\ &(x-5)(x+4)\\ &(x+y-3z)(y+3x)\\ &(3x^2-4x+5)(x-7)\end{aligned}$$

This work also applies to squares of brackets.

$$\begin{aligned}(2x+5)^2 &= (2x+5)(2x+5)\\ &= 4x^2+10x+10x+25\\ &= 4x^2+20x+25.\end{aligned}$$

There is, however a shortcut to squaring brackets which doesn't require writing out the brackets twice and multiplying them.

Rule 1.32 — Squaring Brackets with Two Terms. For any terms a, b,

$$(a+b)^2 = a^2+2ab+b^2.$$

Put less formally, when squaring a bracket with two terms its fully expanded form is the sum of the square of each term and twice their product.

Example 1.33 Fully expand $(2x+5)^2$ using Rule 1.32.

$$\begin{aligned}(2x+5)^2 &= (2x)^2 + 2(2x)(5) + 5^2 \\ &= 4x^2 + 20x + 25.\end{aligned}$$

Example 1.34 Fully expand $(3x-5)^2$ using Rule 1.32.

$$\begin{aligned}(3x-5)^2 &= (3x)^2 + 2(3x)(-5) + (-5)^2 \\ &= 9x^2 - 30x + 25.\end{aligned}$$

Note 1.35 Notice in Example 1.34 how the negative sign comes with the 5, and when -5 is squared it becomes positive. As a shortcut, when squaring a bracket with two terms, if both terms are positive every term in the expansion is positive, and if one term is negative the "middle term" is negative.

In the rare case that both terms are negative, all terms end up being positive.

Example 1.36 Fully expand $(-2x-5)^2$ using Rule 1.32.

$$\begin{aligned}(-2x-5)^2 &= (-2x)^2 + 2(-2x)(-5) + (-5)^2 \\ &= 4x^2 + 20x + 25.\end{aligned}$$

Question 1.37 Fully expand out the following expressions.

$$\begin{gathered}(3x+4)^2 \\ (5x-7)^2 \\ (3x+y)^2 \\ (5x-2y^2)^2\end{gathered}$$

In the case of squaring a bracket with three terms, one can revert back to writing both brackets side by side, or use the following rule.

Rule 1.38 — Squaring Brackets with Three Terms. For any terms a, b, c,

$$(a+b+c)^2 = a^2 + b^2 + c^2 + 2ab + 2ac + 2bc.$$

Notice that each term is squared, and each pair of terms is multiplied by 2.

Example 1.39 Fully expand the expression $(2x+5y-3xz)^2$.

Method 1: We will just write two brackets side by side.

$$\begin{aligned}(2x+5y-3xz)^2 &= (2x+5y-3xz)(2x+5y-3xz)\\ &= 4x^2+10xy-6x^2z+10xy+25y^2-15xyz-6x^2z-15xyz+9x^2z^2\\ &= 4x^2+20xy-12x^2z+25y^2-30xyz+9x^2z^2.\end{aligned}$$

Method 2: We will use Rule 1.38.

$$\begin{aligned}(2x+5y-3xz)^2 &= (2x)^2+(5y)^2+(-3xz)^2+2(2x)(5y)+2(2x)(-3xz)+2(5y)(-3xz)\\ &= 4x^2+25y^2+9x^2y^2+20xy-12x^2z-30xyz.\end{aligned}$$

Notice that these answers are the same, the terms are just in a different order.

Note 1.40 When squaring a term that has a minus sign, since we know the minus sign goes away there is no need to include it. In Example 1.39 in particular, I didn't need to write $(-3xz)^2$, I could have just written $(3xz)^2$. I will do this regularly throughout the notes.

Question 1.41 Fully expand the following expressions.

$$\begin{gathered}(2x+y+3z)^2\\ (3xy-4x+z)^2\\ (2x-3x^2-5x^3)^2\end{gathered}$$

When expanding cubes, we can just write the three brackets side by side, and multiply them together two at a time.

Example 1.42 Fully expand $(3x+5)^3$.

$$\begin{aligned}(3x+5)^3 &= (3x+5)(3x+5)(3x+5)\\ &= (3x+5)(9x^2+15x+15x+25)\\ &= (3x+5)(9x^2+30x+25)\\ &= 27x^3+90x^2+75x+45x^2+150x+125\\ &= 27x^3+135x^2+225x+125.\end{aligned}$$

Notice how, when we multiply the second and third bracket together, the entire answer should get multiplied by the first $(3x+5)$ brakcet. Therefore it goes in a bracket itself.

Question 1.43 Fully expand the following expressions.

$$\begin{gathered}(2x+4)^3\\ (1-2x)^3\end{gathered}$$

We will learn about how to expand brackets taken to higher powers in Section 1.12.

1.8 Factorising Four Terms By Grouping

1.8.1 $++++$ & $+-+-$ Expressions

Now that we can show that

$$\begin{aligned}(7+a)(x+y) &= 7(x+y)+a(x+y)\\ &= 7x+7y+ax+ay,\end{aligned}$$

how do we do the reverse? The answer is, again, that we factorise. Given four terms, in simple cases we can factorise the first two terms and the second two terms separately (we will worry about what "non-simple" cases are later). In this case, starting with

$$7x+7y+ax+ay,$$

we can factorise the first two terms and the second two terms to get

$$7(x+y)+a(x+y).$$

Now, notice that both terms have an $(x+y)$ factor (it may be strange to consider these as terms as they have an addition symbol, but we are considering $(x+y)$ as a single factor of the expression as it is in brackets). If instead we had

$$7z+az,$$

we could write

$$7z+az = z(7+a).$$

There is no reason we can't do this factorisation if z is replaced with the bracketed $(x+y)$.

$$\begin{aligned}7x+7y+ax+ay &= 7(x+y)+a(x+y)\\ &= (x+y)(7+a).\end{aligned}$$

Notice that these steps are the same as those at the beginning of the section but in reverse. When factoring out the $(x+y)$ term in the last line it can be written on the right or the left, it doesn't matter. This final answer is **fully factorised** as the expression is written as factors that cannot be factored further.

Example 1.44 Fully factorise $4xy+7y+8x+14$.

$$\begin{aligned}4xy+7y+8x+14 &= y(4x+7)+2(4x+7)\\ &= (4x+7)(y+2).\end{aligned}$$

To see an example with minus signs, consider the following.

Example 1.45 Fully factorise $3x^2y-12xy+5xz-20z$.

$$\begin{aligned}3x^2y-12xy+5xz-20z &= 3xy(x-4)+5z(x-4)\\ &= (x-4)(3xy+5z).\end{aligned}$$

Notice in both examples that in order to complete the final step both brackets must be equal after the first round of factorising. If this is not the case then there is either a mistake in calculation or the factor removed was not the HCF. As a small note, we never have a reason to factor out anything other than the HCF, so when we say "factorise" we specifically mean factoring out the HCF.

Question 1.46 Fully factorise the following expressions.

$$\begin{aligned}
&3x+6y+xz+2yz\\
&6ab-9a^2b+4x-6ax\\
&5x^2y^3+15xyz+8xy^2z+24z^2\\
&7x+7+xy+x^2y\\
&12x^2-4x+15xy-5y
\end{aligned}$$

1.8.2 $++--$ & $+--+$ Expressions

We will now consider cases that are not as straightforward. Most of these examples can be summarised as having "awkward minus signs".

Example 1.47 Factorise $5x+5y-ax-ay$.

If we try to factorise $5x+5y-ax-ay$ as

$$5x+5y-ax-ay=5(x+y)+a(-x-y),$$

we see that we don't have the same bracket in both terms, and so can't proceed any further. What if we also factored out the minus sign? Then

$$\begin{aligned}
5x+5y-ax-ay&=5(x+y)-a(x+y)\\
&(x+y)(5-a).
\end{aligned}$$

In conclusion, sometimes to get the same expression in both brackets we have to factor a minus sign out of the second pair of terms.

Example 1.48 Fully factorise

$$4x-8-xy+2y.$$

$$\begin{aligned}
4x-8-xy+2y&=4(x-2)-y(x-2)\\
&=(x-2)(4-y).
\end{aligned}$$

It may be confusing that we can factor out a negative sign out of the $+2y$ term. Remember that two negative signs give a plus sign, so it may help to imagine $+2y=--2y$, i.e. $2y$ has two minus signs and we factor out one. This is a good time to remember that you know your factorising is correct if expanding out the brackets gets you back where you started. It is true that

$$-y(x-2)=-xy+2y,$$

so we know this is a correct factorisation.

To summarise our dealings with signs so far, if when factorising one of the brackets is the negative of the other (i.e. all terms inside one bracket have the opposite sign to the same term in the other), then a minus sign should be factored out of one of the pairs.

Question 1.49 Fully factorise the following expressions.

$$27xy - 72x - 12y + 32$$
$$x^3z + 2x^2y - 4xyz - 8y^2$$
$$5b + 10a - 4a^2b - 8a^3$$
$$pq^3 - p^2qr - 3q^2r + 3pr^2$$

1.8.3 1 and -1 as a HCF

We have two more cases to consider, both of which centre around difficulty in factorising.

Example 1.50 Factorise $8x + 12xy + 2 + 3y$.

$$8x + 12xy + 2 + 3y = 4x(2 + 3y) + 2 + 3y.$$

We can factor out a $4x$ from the first pair of terms but cannot factor anything out of the second pair. However the second pair is exactly the bracket we're looking for. So we simply factor out the HCF, which is 1, as normal to get

$$\begin{aligned} 8x + 12xy + 2 + 3y &= 4x(2 + 3y) + 1(2 + 3y) \\ &= (2 + 3y)(4x + 1). \end{aligned}$$

Example 1.51 Factorise $9a^2 + 3ab^2 - 3a - b^2$.

$$\begin{aligned} 9a^2 + 3ab^2 - 3a - b^2 &= 3a(3a + b^2) - (3a + b^2) \\ &= (3a + b^2)(3a - 1). \end{aligned}$$

Notice we had to also factor out the minus sign when factorising $-3a - b^2$ so that both brackets were $(3a + b^2)$.

Example 1.52 Factorise $3pr - 2qr - 3p + 2q$.

$$\begin{aligned} 3pr - 2qr - 3p + 2q &= r(3p - 2q) - (3p - 2q) \\ &= (3p - 2q)(r - 1). \end{aligned}$$

Again, we had to factor out the minus sign when factorising $-3p + 2q$ to make both brackets $(3p - 2q)$, and factorising the minus sign out of $+2q$ made the leftover term $-2q$.

Question 1.53 Fully factorise the following expressions.

$$6x^2 + 4xy + 3x + 2y$$
$$11ac - 22bc - a + 2b$$
$$3p + q - 12p^2qr - 4pq^2r$$

1.8.4 Rearranging Terms

Finally, what if we cannot factor either of the pairs of terms, or if the brackets we make are nothing alike?

Example 1.54 Fully factorise $7x+ay+7y+ax$.

We can't factor the first or second pair of terms. This is because the terms were given to us in the wrong order. This can be fixed by swapping either of the first two terms with either of the second two terms. Because of this, for simplicity you can **always** fix this issue by swapping the **second and third** term. In this case,

$$\begin{aligned} 7x+ay+7y+ax &= 7x+7y+ay+ax \\ &= 7(x+y)+a(y+x) \\ &= (x+y)(7+a). \end{aligned}$$

Question 1.55 Fully factorise the following expressions by swapping terms.

$$\begin{aligned} &3x+2yz+6y+xz \\ &12x^2-5y-4x+15xy \\ &6x^2+2y+4xy+3x \\ &5b-8a^3+10a-4a^2b \end{aligned}$$

1.8.5 Deciding Which Approach to Take

In conclusion, depending on the nature of the four terms our factorising by grouping is slightly different. Part of the trick in tackling these problems is knowing which of the types of problems we studied today that they are.

Question 1.56 Fully factorise the following expressions.

$$\begin{aligned} &3x^2-5xz+6xy-10yz \\ &a^3-2a^2b+3ab^2-6b^3 \\ &x^3-x^2y+xy^2-y^3 \\ &4x^3-12x^2y-9xy^2+27y^3 \\ &2x^3+4x^2y-5xy^2-10y^3 \\ &3a^4-9a^3b+4ab^2-12b^3 \\ &x^4+40xy^3-5x^3y-8x^2y^2 \\ &4x^4+16x^3y-xy^2-4y^3 \end{aligned}$$

1.9 Factorising Quadratics

1.9.1 Of the Form x^2+bx+c

We can see from earlier work in this chapter that

$$\begin{aligned} (x+2)(x+3) &= x^2+3x+2x+6 \\ &= x^2+5x+6. \end{aligned}$$

However given the quadratic x^2+5x+6, how can we factor it back to $(x+2)(x+3)$? This is easier if, instead of x^2+5x+6, we are given

$$x^2+3x+2x+6$$

as we can apply skills we learned in Section 1.8:

$$\begin{aligned} x^2+3x+2x+6 &= x(x+3)+2(x+3) \\ &= (x+3)(x+2). \end{aligned}$$

However the $3x$ and the $2x$ have combined, and if we just have $5x$ we wouldn't know to split it into $3x+2x$.

To overcome this disadvantage, consider where the 5 and 6 come from in the calculation

$$\begin{aligned} (x+2)(x+3) &= x^2+3x+2x+6 \\ &= x^2+5x+6. \end{aligned}$$

The 2 and the 3 **add** to give 5 and **multiply** to give 6.

Example 1.57 Factorise $x^2+7x+12$.

If we want to write

$$x^2+7x+12 = (x+\_\_)(x+\_\_),$$

the numbers in the spaces should add to give 7 and multiply to give 12. The numbers that do this are 3 and 4, so

$$x^2+7x+12 = (x+3)(x+4).$$

As the brackets are being multiplied order doesn't matter; we can write the answer as

$$x^2+7x+12 = (x+4)(x+3)$$

if we want.

We can therefore factorise quadratics with an x^2 term by finding two numbers that add to give the coefficient of the x term and multiply to give the constant term, if they exist. Sometimes they don't exist, and sometimes we don't have an x^2 term (for example, $3x^2+5x+4$). We will deal with both of these cases later.

Sometimes it can be hard to sort through all possible numbers in your head. If two numbers don't come to mind quickly it may be better to write down all possibilities.

Example 1.58 Factorise $x^2+21x+108$.

If we can't think of two numbers that add to give 21 and multiply to give 108, we could write down all pairs of numbers that multiply to give 108:

$$
\begin{aligned}
&1,\ 108\\
&2,\ 54\\
&3,\ 36\\
&4,\ 27\\
&6,\ 18\\
&9,\ 12\\
&\cancel{12,\ 9}
\end{aligned}
$$

This is done by first dividing 108 by 1, and increasing the number on the left, skipping all numbers that don't divide into 108. We then stop at the pair 12, 9, as at that point we're just doubling back on ourselves. Now that we're looking at all the pairs rather than thinking of them in our head, it's easier to see that the pair we want is 9, 12. There is a reason we first look at all the pairs that multiply to give the constant term rather than all the pairs that add to the x coefficient; there are always much fewer of the former.

Question 1.59 Factorise each of the following quadratics.

$$
\begin{aligned}
&x^2+8x+15\\
&x^2+11x+24\\
&x^2+6x+9\\
&x^2+11x+10\\
&x^2+48x+135
\end{aligned}
$$

What if some of the signs are negative?

Example 1.60 Factorise x^2-6x+8.

We proceed as before, allowing for negative numbers. Two numbers that multiply to give 8 and add to give -6 are -2 and -4:

$$
\begin{aligned}
-2\times-4 &= +8\\
-2+(-4) &= -2-4\\
&= -6.
\end{aligned}
$$

Therefore

$$x^2-6x+8=(x-2)(x-4).$$

If the constant coefficient is positive then both terms must be the same sign. However if they are the same sign, and the x coefficient is negative, they must both be negative.

Example 1.61 Factorise $x^2+3x-10$.

In this case the two numbers must be of opposite sign to multiply to give -10. If they add to give $+3$ then the larger number must be positive. These numbers are 5 and -2, so that

$$x^2 + 3x - 10 = (x+5)(x-2).$$

Example 1.62 Factorise $x^2 - 4x - 5$.

The two numbers must still be of opposite sign to multiply to give -5, but now the larger number must be negative to add to give -4. These numbers are -5 and $+1$, so that

$$x^2 - 4x - 5 = (x-5)(x+1).$$

Again we can do better than just guessing pairs of numbers. Given any quadratic of the form $x^2 + bx + c$, we can write down all pairs of positive numbers that multiply to give the positive version of the constant term. Consider the following example.

Example 1.63 Factorise $x^2 - 3x - 108$.

We first write down all pairs of positive numbers that multiply to give $+108$, as before.

$$\begin{gathered} 1,\ 108 \\ 2,\ 54 \\ 3,\ 36 \\ 4,\ 27 \\ 6,\ 18 \\ 9,\ 12 \\ \cancel{12,\ 9} \end{gathered}$$

We then consider the signs. Because the constant term is negative exactly one of the pair of numbers is negative. Because the coefficient of the x term is negative the larger number is negative. If you look at the way we construct our pairs the smaller of the pair is always on the left, so we can form our "candidate" pairs by adding minus signs to all numbers on the right.

$$\begin{gathered} 1,\ -108 \\ 2,\ -54 \\ 3,\ -36 \\ 4,\ -27 \\ 6,\ -18 \\ 9,\ -12. \end{gathered}$$

We can then see that the pair that adds to -3 is $9, -12$, so that

$$x^2 - 3x - 108 = (x+9)(x-12).$$

Example 1.64 Factorise $x^2 - 23x + 120$.

Again the numbers may be too big to do in our heads, so we write down all the pairs of positive numbers that add to give $+120$.

$$\begin{array}{r} 1,\ 120 \\ 2,\ 60 \\ 3,\ 40 \\ 4,\ 30 \\ 5,\ 24 \\ 6,\ 20 \\ 8,\ 15 \\ 10,\ 12 \\ \cancel{12,\ 10} \end{array}$$

Because the constant coefficient is positive both numbers must be of the same sign. Because the x coefficient is negative they then must both be negative. Therefore we add a minus sign to both numbers.

$$\begin{array}{r} -1,\ -120 \\ -2,\ -60 \\ -3,\ -40 \\ -4,\ -30 \\ -5,\ -24 \\ -6,\ -20 \\ -8,\ -15 \\ -10,\ -12 \end{array}$$

We can then see that the pair should be -8, -15 so that

$$x^2 - 23x + 120 = (x-8)(x-15).$$

Note 1.65 You should notice when trying to try this method of writing down pairs of numbers, regardless of what the sign of the numbers are the **size** of the numbers will always decrease as you go down the list. Look at what happens when we go down the lists in both of the previous examples.

$$\begin{array}{l} 1,\ -108 \to -107 \\ 2,\ -54 \to -52 \\ 3,\ -36 \to -33 \\ 4,\ -27 \to -23 \\ 6,\ -18 \to -12 \\ 9,\ -12 \to -3 \end{array} \qquad \begin{array}{r} -1,\ -120 \to -121 \\ -2,\ -60 \to -62 \\ -3,\ -40 \to -43 \\ -4,\ -30 \to -34 \\ -5,\ -24 \to -29 \\ -6,\ -20 \to -26 \\ -8,\ -15 \to -23 \\ -10,\ -12 \to -22 \end{array}$$

Therefore you don't have to check every pair; you should have a rough idea where in the list your pair should be, and you can go up and down the list depending on whether you guess too high or low a number.

Question 1.66 Factorise each of the following quadratics.

$$x^2 - 12x + 20$$
$$x^2 - 3x - 28$$
$$x^2 + 3x - 40$$
$$x^2 - 13x + 12$$
$$x^2 - 37x + 160$$

1.9.2 Of the Form $ax^2 + bx + c$, $a \neq 1$

We now must tackle the case of factorising quadratics of the form $ax^2 + bx + c$ where $a \neq 1$, for example

$$3x^2 + 13x + 4.$$

In this case we have a four step process.

Rule 1.67 Given a quadratic of the form
$ax^2 + bx + c$,

1. Multiply a and c.
2. Find a pair of numbers that multiply to give ac and add to give b. Call these numbers p, q.
3. Replace bx with $px + qx$.
4. Factor the four terms as in Section 1.8.

Example 1.68 Factorise $3x^2 + 13x + 4$.

First, we multiply 3 and 4 to get 12. Second, we find two numbers that multiply to give 12 and add to give 13. These numbers are 12 and 1. Third, we separate $13x$ into $12x + x$ to get

$$3x^2 + 13x + 4 = 3x^2 + 12x + x + 4.$$

Fourth, we factorise the four terms as in Section 1.8.

$$\begin{aligned} 3x^2 + 13x + 4 &= 3x^2 + 12x + x + 4 \\ &= 3x(x+4) + (x+4) \\ &= (x+4)(3x+1). \end{aligned}$$

This approach also works for quadratics with negative signs, and it doesn't matter which of the two numbers found in Step 2 is placed first in the new expression for the quadratic. See that if we reverse the order,

$$\begin{aligned}3x^2+13x+4 &= 3x^2+x+12x+4\\ &= x(3x+1)+4(3x+1)\\ &= (3x+1)(x+4),\end{aligned}$$

the same answer as before.

Example 1.69 Factorise $5x^2-3x-2$.

First, $5\times -2=-10$. Two numbers that multiply to give -10 and add to give -3 are -5, $+2$ (you can still use the listing methods from Examples 1.63 and 1.64 here if you can't guess the numbers). Then

$$\begin{aligned}5x^2-3x-2 &= 5x^2-5x+2x-2\\ &= 5x(x-1)+2(x-1)\\ &= (x-1)(5x+2).\end{aligned}$$

Question 1.70 Factorise each of the following quadratics.

$$\begin{gathered}2x^2+7x+6\\ 3x^2-8x+5\\ 7x^2-3x-4\\ 4x^2-4x-15\end{gathered}$$

1.9.3 Of the form ax^2+bx+c, when $b=0$ or $c=0$

We must finally consider the question of factorising ax^2+bx+c when either $c=0$ or $b=0$ ($a=0$ means the expression is not really a quadratic and so won't be considered). The case where $c=0$ can be tackled using tools learned from 1.6. For example

$$\begin{aligned}x^2-7x &= x(x-7),\\ 3x^2+12x &= 3x(x+4).\end{aligned}$$

The case where $b=0$ results in quadratics such as

$$\begin{gathered}x^2-9,\\ 3x^2+15.\end{gathered}$$

We will learn how to factorise these expressions and more in Section 1.10.

1.10 Difference of Two Squares

See that

$$\begin{aligned}(x-3)(x+3) &= x^2+3x-3x-9\\ &= x^2-9.\end{aligned}$$

In general, we have the following rule.

Rule 1.71 — Difference of Two Squares. For any two terms a and b,

$$a^2 - b^2 = (a-b)(a+b).$$

This formula needs to be memorised as it isn't in *Formulae and Tables*.

Example 1.72 Factorise $x^2 - 16$.

$$\begin{aligned} x^2 - 16 &= x^2 - 4^2 \\ &= (x-4)(x+4). \end{aligned}$$

It is important that each term is written as a square before factorising.

Example 1.73 Factorise $4x^2 - 25$.

$$\begin{aligned} 4x^2 - 25 &= (2x)^2 - 5^2 \\ &= (2x-5)(2x+5). \end{aligned}$$

Example 1.74 Factorise $121x^4 - 4$.

When writing each term as a square, each component can be considered separately, and what goes inside the bracket is the square root of what was in the original term. As $\sqrt{x^4} = x^2$ and $\sqrt{121} = 11$,

$$\begin{aligned} 121x^4 - 4 &= (11x^2)^2 - 2^2 \\ &= (11x^2 - 2)(11x^2 + 2). \end{aligned}$$

Example 1.75 Factorise $49x^2y^4 - 1$.

$$\begin{aligned} 9x^2y^4 - 1 &= (3xy^2)^2 - 1^2 \\ &= (3xy^2 - 1)(3xy^2 + 1). \end{aligned}$$

Question 1.76 Factorise each of the following expressions.

$$4 - x^2$$
$$a^2 - 9b^2$$
$$36a^2b^2 - 4p^2$$
$$x^4 - 49y^6z^2$$

Returning to quadratics with no x term, we can now factorise a quadratic if the constant term is negative. For example,

$$\begin{aligned} x^2 - 16 &= (x-4)(x+4), \\ 4x^2 - 9 &= (2x)^2 - 3^2 = (2x-3)(2x+3) \end{aligned}$$

Question 1.77 Factorise the following quadratics.

$$x^2 - 9$$
$$25x^2 - 81$$
$$4x^2 - 1$$
$$x^2 + 5x$$
$$12x^2 - 9x$$

On the other hand we cannot factorise quadratics with no x term and a positive constant term, for example $x^2 + 9$. The reasons for this will become more clear when we study quadratics again in Chapter 3.

1.11 *Sum and Difference of Two Cubes

Just like we have the formula

$$a^2 - b^2 = (a-b)(a+b),$$

we have two formulas for cubes; one for the sum and one for the difference.

Rule 1.78 — Sum and Difference of Two Cubes. For any two terms a and b,

$$a^3 + b^3 = (a+b)(a^2 - ab + b^2),$$
$$a^3 - b^3 = (a-b)(a^2 + ab + b^2).$$

These formulae need to be memorised as they aren't in *Formulae and Tables*. Much of the work in this section is similar to our previous work on squares; we first write both terms as a cube and then use Rule 1.78. When writing a term as a cube, take the cube root of the coefficient and divide any power of an algebraic component by 3.

Example 1.79 Factorise $8x^3 - 27$.

$$\begin{aligned} 8x^3 - 27 &= (2x)^3 - 3^3 \\ &= (2x-3)((2x)^2 + (2x)(3) + 3^2) \\ &= (2x-3)(4x^2 + 6x + 9) \end{aligned}$$

Example 1.80 Factorise $64x^3y^6 + 1$.

$$\begin{aligned} 64x^3y^6 + 1 &= (4xy^2)^3 + 1^3 \\ &= (4xy^2 + 1)((4xy^2)^2 - 4xy^2(1) + 1^2) \\ &= (4xy^2 + 1)(16x^2y^4 - 4xy^2 + 1). \end{aligned}$$

Question 1.81 Factorise each of the following expressions.

$$8-27x^3$$
$$125y^3+x^6$$
$$27x^9y^3-1$$
$$x^9-64y^3$$
$$8y^6z^6-27x^3$$

1.12 *Binomial Expansions

In Section 1.7 we learned how to expand brackets of the form $(x+y)^2$, and even $(x+y)^3$ by writing it as

$$\begin{aligned}(x+y)^3 &= (x+y)(x+y)^2\\ &= (x+y)(x^2+2xy+y^2)\\ &= x^3+2x^2y+xy^2+x^2y+2xy^2+y^3\\ &= x^3+3x^2y+3xy^2+y^3.\end{aligned}$$

However this is time consuming and if we were to apply it to expanding an expression like $(x+y)^6$ it would take even longer. We will learn how to quickly expand such brackets but before we do we must consider some new notation. This notation is also used in probability (which we will not cover this year).

The notation $\begin{pmatrix} n \\ r \end{pmatrix}$, or nCr as you will see on your calculator, reads as "n choose r". It actually represents the number of possible teams of size r that can be made from a group of n people, and so when n and r are replaced with numbers it becomes a number. There are ways of calculating it by hand when n and r are whole numbers but we will not get into it that here and instead rely almost wholly on our calculators. For example, $\begin{pmatrix} 7 \\ 3 \end{pmatrix}$ can be calculated immediately on your calculator by pressing 7, then nCr, then 3, to get an answer of $\begin{pmatrix} 7 \\ 3 \end{pmatrix} = 35$.

Question 1.82 Use your calculator to find $\begin{pmatrix} 8 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 8 \\ 6 \end{pmatrix}$.

You should have found that your answers in Question 1.82 were the same. This is because $6+2=8$ and is always the case; in fact for $\begin{pmatrix} n \\ r \end{pmatrix}$ we have the following three rules.

Rule 1.83 — Rules for nCr. For any n, r,

$$\begin{pmatrix} n \\ r \end{pmatrix} = \begin{pmatrix} n \\ n-r \end{pmatrix},$$
$$\begin{pmatrix} n \\ 1 \end{pmatrix} = \begin{pmatrix} n \\ n-1 \end{pmatrix} = n,$$
$$\begin{pmatrix} n \\ 0 \end{pmatrix} = \begin{pmatrix} n \\ n \end{pmatrix} = 1.$$

So for example,

$$\begin{aligned}
\binom{8}{3} &= \binom{8}{5}, \\
\binom{8}{2} &= \binom{8}{6}, \\
\binom{8}{1} &= \binom{8}{7} = 8, \\
\binom{8}{0} &= \binom{8}{8} = 1.
\end{aligned}$$

Now that we have covered notation we are ready to learn about the Binomial Theorem, which allows us to quickly expand expressions like $(x+y)^n$ when n is large.

Theorem 1.84 — The Binomial Theorem (pg. 20 of *Formulae & Tables*). For any $n \geq 1$,

$$(x+y)^n = \binom{n}{0}x^n + \binom{n}{1}x^{n-1}y + \ldots + \binom{n}{r}x^{n-r}y^r + \ldots + \binom{n}{n-1}xy^{n-1} + \binom{n}{n}y^n.$$

The theorem appears in the following form on page 20 of *Formulae & Tables*.

An Teoirim dhéthéarmach **Binomial theorem**

$$(x+y)^n = \sum_{r=0}^{n}\binom{n}{r}x^{n-r}y^r = \binom{n}{0}x^n + \binom{n}{1}x^{n-1}y + \binom{n}{2}x^{n-2}y^2 + \cdots + \binom{n}{r}x^{n-r}y^r + \cdots + \binom{n}{n}y^n$$

comhéifeachtaí déthéarmacha $\binom{n}{r} = {}^nC_r = C(n,r) = \dfrac{n!}{r!(n-r)!}$ binomial coefficients

Figure 1.2

Example 1.85 Fully expand $(x+y)^4$.

$$\begin{aligned}
(x+y)^4 &= \binom{4}{0}x^4 + \binom{4}{1}x^3y + \binom{4}{2}x^2y^2 + \binom{4}{3}xy^3 + \binom{4}{4}y^4 \\
&= x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4.
\end{aligned}$$

Notice that you should only need to use your calculators here to calculate $\binom{4}{2}$; the rest are immediate from Rule 1.83.

Question 1.86 Fully expand

$$(x+y)^5$$

What if we want to expand a more complicated expression?

Example 1.87 Fully expand $(4x+2)^4$.

We replace x and y with $4x$ and 2 in the formulation of Theorem 1.84. We take the entirety of each term to the required power.

$$
\begin{aligned}
(4x+2)^4 &= \binom{4}{0}(4x)^4 + \binom{4}{1}(4x)^3(2) + \binom{4}{2}(4x)^2(2)^2 + \binom{4}{3}(4x)(2)^3 + \binom{4}{4}2^4 \\
&= 256x^4 + (4)(64x^3)(2) + 6(16x^2)(4) + 4(4x)(8) + 16 \\
&= 256x^4 + 512x^3 + 384x^2 + 128x + 16.
\end{aligned}
$$

Example 1.88 Fully expand $(2x-3)^3$.

Here the minus goes everywhere the 3 does, including inside the powers.

$$
\begin{aligned}
(2x-3)^3 &= \binom{3}{0}(2x)^3 + \binom{3}{1}(2x)^2(-3) + \binom{3}{2}(2x)(-3)^2 + \binom{3}{3}(-3)^3 \\
&= 8x^3 + 3(4x^2)(-3) + 3(2x)(9) + (1)(-27) \\
&= 8x^3 - 36x^2 + 54x - 27.
\end{aligned}
$$

Question 1.89 Fully expand the following expressions.

$$
\begin{aligned}
&(2x+3)^5 \\
&(3x-2)^4 \\
&(3x+1)^3 \\
&(2x-1)^4
\end{aligned}
$$

Sometimes we may only be interested in a specific term of the binomial expansion. For example, say we are only interested in the coefficient in front of the x^7 term in $(4x+1)^9$. Then we are interested in

$$
\binom{9}{r}(4x)^{9-r}1^r
$$

when the power over the x term is equal to 7. So the term is

$$
\begin{aligned}
\binom{9}{2}(4x)^7 1^2 &= 36(4^7)x^7 \\
&= 589824x^7.
\end{aligned}
$$

So the coefficient is 589,824. Notice when constructing the term the power of the x term is the power we want, the two powers always add to the top number in the brackets and the number at the bottom in the brackets is equal to one of the powers (it doesn't matter which one).

Example 1.90 Find the coefficient in front of the x^3 term in the expansion of $(2-3x)^8$.

All terms look like

$$
\binom{8}{\ }(2)^{\square}(-3x)^{\square}
$$

and if we want the x^3 term then the term we care about looks like

$$\begin{pmatrix} 8 \\ \end{pmatrix} (2)^{\square}(-3x)^3$$

The powers add up to 8 so our term looks like

$$\begin{pmatrix} 8 \\ \end{pmatrix} (2)^5(-3x)^3$$

Finally the lower number in the $\begin{pmatrix} 8 \\ \end{pmatrix}$ term can be let equal to 5 or 3, it doesn't matter. Then we can start calculating:

$$\begin{aligned} \begin{pmatrix} 8 \\ 3 \end{pmatrix} (2)^5(-3x)^3 &= (56)(32)(-27x^3) \\ &= -48,384x^3. \end{aligned}$$

Question 1.91 Find the coefficient of the x^3 term in the following binomial expansions.

$$(3+2x)^8$$
$$(7-x)^6$$

Question 1.92 Find the coefficient of the x^6 term in the following binomial expansions.

$$(3+2x)^8$$
$$(7-x)^6$$

We can find the constant coefficient of a binomial expansion by understanding that the constant coefficient is the one associated with $x^0 = 1$ (remember we treat an absent x as the same as x^0).

Example 1.93 Find the constant coefficient in the expansion of $(2-3x)^8$.

All terms look like

$$\begin{pmatrix} 8 \\ \end{pmatrix} (2)^{\square}(-3x)^{\square}$$

We want the $-3x$ to be to the power of 0, so that the term we should be looking at is

$$\begin{aligned} \begin{pmatrix} 8 \\ 0 \end{pmatrix} (2)^8(-3x)^0 &= (1)(256)(1) \\ &= 256. \end{aligned}$$

Question 1.94 Find the constant coefficient in the following binomial expansions.

$$(3+2x)^8$$
$$(7-x)^6$$

1.13 Summary

Expanding and factorising brackets are not usually standalone questions on Leaving Cert papers in Higher or Ordinary Level. Instead, they serves as a foundation, as expanding brackets and factorising expressions are necessary as tools when tackling larger problems. Binomial expansions have occasionally appeared as questions by themselves on the Higher Level paper.

1.14 Homework

Factorising Using the HCF

1. Factorise each of the following expressions.
   (a) x^2+5x
   (b) $5x^2-20x$
   (c) $4b^2-12ab^3$
   (d) $10p^2q+5pq^2$
   (e) $21xy^2+35x^3y^2$
   (f) $7x^9y^6+12x^4y^{12}$
   (g) $2a^2b-4ab^2+12abc$
   (h) $12p^2q^3r-15pqr+21qr^2$
   (i) $5x^3-10x^2+15x$
   (j) $4x^2-6xy+8xz$

Factorising Four Terms By Grouping

2. Fully factorise each of the following expressions.
   (a) $x^2+ax+3x+3a$
   (b) $6ax^2+9a+8x^2+12$
   (c) $12a^2+8ab+9ac+6bc$
3. Fully factorise each of the following expressions.
   (a) $5xy-10y^2+4xz-8yz$
   (b) $2x^2-4cd+5x^2y-10cdy$
   (c) $3abx^2-3bxy+5axy^2-5y^3$
4. Fully factorise each of the following expressions.
   (a) $5a-10ay-4x^2+8x^2y$
   (b) $x^2+4xy-3ax-12ay$
   (c) $6ax^2+9a-8x^2-12$
   (d) $10ab-5ac-2bd+cd$
5. Fully factorise each of the following expressions.
   (a) $3pq^2+6p^2q+q+2p$
   (b) $5xy+3y-5x-3$
   (c) $a^2+bc+ab+ac$
   (d) $6x^2-9xy+6ay-4ax$
6. Fully factorise each of the following expressions.
   (a) $ax+2ay+bx+2by$
   (b) $6a^2-10bc+4ac-15ab$
   (c) $2ax-6ay+x-3y$
   (d) $6a^2c-6ab-4bc+9a^3$
   (e) $12xy-21x-8y+14$

Factorising Quadratics

7. Factorise each of the following expressions.
   (a) $x^2+9x+20$
   (b) $x^2+14x+24$
   (c) x^2+6x+5
8. Factorise each of the following expressions.
   (a) $x^2-11x+28$
   (b) $x^2-11x+10$
   (c) $x^2-14x+33$
9. Factorise each of the following expressions.
   (a) $x^2+3x-28$
   (b) $x^2-3x-28$
   (c) $x^2-6x-55$
   (d) $x^2+4x-21$

10. Factorise each of the following expressions.

    (a) $2x^2+11x+14$

    (b) $5x^2-12x-9$

    (c) $8x^2-14x+3$

    (d) $9x^2-x-10$

Difference of Two Squares

11. Factorise the following expressions.

    (a) x^2-16y^2

    (b) $36-121y^2$

    (c) $1-81y^2$

    (d) x^2y^2-16

    (e) $81h^2k^4-25p^6q^2$

*Sum and Difference of Two Cubes

12. Factorise the following expressions.

    (a) x^3-8y^3

    (b) $27-64y^3$

    (c) $1+y^6$

    (d) $x^3y^3-27z^3$

    (e) $64p^3q^6+27x^3$

*Binomial Expansions

13. Fully expand the following expressions using The Binomial Theorem.

    (a) $(x+y)^6$

    (b) $(1+x)^4$

    (c) $(4-3x)^5$

    (d) $(3x+2y)^4$

14. For each of the following expressions, find the coefficient of the x^n term where n is given in the question.

(a)	$(5+2x)^9$	x^2
(b)	$(3-4x)^8$	x^4
(c)	$(1-3x)^9$	x^9
(d)	$(2-3x)^6$	constant term

1.15 Homework Solutions

Factorising Using the HCF

1. (a) $x(x+5)$
   (b) $5x(x-4)$
   (c) $4b^2(1-3ab)$
   (d) $5pq(2p+q)$
   (e) $7xy^2(3+5x^2)$
   (f) $x^4y^6(7x^5+12y^6)$
   (g) $2ab(a-2b+6c)$
   (h) $3qr(4p^2q^2-5p+7r)$
   (i) $5x(x^2-2x+3)$
   (j) $2x(2x-3y+4z)$

Factorising Four Terms By Grouping

2. (a) $(x+3)(x+a)$
   (b) $(3a+4)(2x^2+3)$
   (c) $(4a+3c)(3a+2b)$
3. (a) $(5y+4z)(x-2y)$
   (b) $(2+5y)(x^2-2cd)$
   (c) $(3bx+5y^2)(ax-y)$
4. (a) $(5a-4x^2)(1-2y)$
   (b) $(x-3a)(x+4y)$
   (c) $(3a-4)(2x^2+3)$
   (d) $(5a-d)(2b-c)$
5. (a) $(3pq+1)(q+2p)$
   (b) $(y-1)(5x+3)$
   (c) $(a+b)(a+c)$
   (d) $(2x-3y)(3x-2a)$
6. (a) $(a+b)(x+2y)$
   (b) $(2a-5b)(3a+2c)$
   (c) $(x-3y)(1+2a)$
   (d) $(3a^2-2b)(2c+3a)$
   (e) $(3x-2)(4y-7)$

Factorising Quadratics

7. Factorise each of the following expressions.
   (a) $(x+4)(x+5)$
   (b) $(x+12)(x+2)$
   (c) $(x+5)(x+1)$
8. Factorise each of the following expressions.
   (a) $(x-7)(x-4)$
   (b) $(x-1)(x-10)$
   (c) $(x-11)(x-3)$
9. Factorise each of the following expressions.
   (a) $(x-4)(x+7)$
   (b) $(x+4)(x-7)$
   (c) $(x-11)(x+5)$
   (d) $(x-3)(x+7)$
10. Factorise each of the following expressions.
   (a) $(2x+7)(x+2)$
   (b) $(5x+3)(x-3)$
   (c) $(2x-3)(4x-1)$
   (d) $(9x-10)(x+1)$

Difference of Two Squares

11. (a) $(x-4y)(x+4y)$
   (b) $(6-11y)(6+11y)$
   (c) $(1-9y)(1+9y)$

(d) $(xy-4)(xy+4)$

(e) $(9hk^2-5p^3q)(9hk^2+5p^3q)$

*Sum and Difference of Two Cubes

12. (a) $(x-2y)(x^2+2xy+4y^2)$

(b) $(3-4y)(9+12y+16y^2)$

(c) $(1+y^2)(1-y^2+y^4)$

(d) $(xy-3z)(x^2y^2+3xyz+9z^2)$

(e) $(4pq^2+3x)(16p^2q^4-12pq^2x+9x^2)$

*Binomial Expansions

13. (a) $x^6+6x^5y+15x^4y^2+20x^3y^3+15x^2y^4+6xy^5+y^6$

(b) $1+4x+6x^2+4x^3+x^4$

(c) $1,024-3,840x+5,760x^2-4,320x^3+1,620x^4-243x^5$

(d) $81x^4+216x^3y+216x^2y^2+96xy^3+16y^4$

14. (a) 11,250,000

(b) 1,451,520

(c) $-19,683$

(d) 64

1.16 Revision

Factorising Using the HCF

Fully factorise the following expressions.

$3x^2 - 9x$

$5x^3 + 10x^2$

$8x^3yz - 6xy^2z$

$8x^3y + 16xy^2 + 12xy$

$6x^3yz - 9xyz + 3xy^2$

Factorising Four Terms By Grouping

Fully factorise the following expressions.

$xy + 5x + 2y + 10$

$3x^2 - 4x + 3xy - 4y$

$3x^2y - 4x^2y^2 - 18z + 24yz$

$3ac - 12bd + 9bc - 4ad$

$3wy^2 - 12xy - wy + 4x$

$15a^3 - 9ab + 10a^2b^2 - 6b^3$

$6x^2 - 10x^4 + 3x - 5x^3$

$3xy + 5y^2 - 3x - 5y$

$49xy - 8 - 14y + 28x$

Factorising Quadratics

Fully factorise the following expressions.

$x^2 + 8x + 12$

$x^2 - 14x + 24$

$x^2 - 4x + 4$

$x^2 + 2x - 15$

$x^2 - 8x - 20$

$x^2 - 2x - 15$

$x^2 - 7x + 10$

$x^2 - 8x + 15$

$x^2 + 17x + 60$

$x^2 + 15x - 54$

$x^2 - 25x + 156$

$3x^2 + 8x + 5$

$4x^2 - 9x + 5$

$5x^2 - 7x - 6$

$3x^2 - x - 2$

$4x^2 + 12x + 9$

$5x^2 - 2x - 3$

$6x^2 - 11x + 4$

Difference of Two Squares

Fully factorise the following expressions.

$x^2 - 9$

$4x^2 - 36$

$9x^4y^2 - z^8$

$16x^6 - 1$

$121 - 100x^{10}$

*Sum and Difference of Two Cubes

Fully factorise the following expressions.

$x^3 - 8$

$8x^3 + 216$

$27x^6y^3 - z^9$

$64x^6 + 1$

$125 - 343x^3$

*Binomial Expansions

Fully expand the following expression using the Binomial Theorem.

$(x+y)^7$

$(4x+1)^4$

$(3x+2)^3$

$(4x-2)^5$

$(2x-5)^4$

In the following expressions, find the coefficient of the given term if the expansion were to be expanded out.

$(x+1)^9$	x^4
$(3x+2)^8$	x^6
$(2x-3)^7$	x^2
$(5x-1)^6$	x
$(3x+2)^9$	constant term

1.17 Revision Solutions

Factorising Using the HCF

$3x(x-3)$
$5x^2(x+2)$
$2xyz(4x^2-3y)$
$4xy(2x^2+4y+3)$
$3xy(2x^2z-3z+y)$

Factorising Four Terms By Grouping

$(x+2)(y+5)$
$(3x-4)(x+y)$
$(x^2y-6z)(3-4y)$
$(a+3b)(3c-4d)$
$(wy-4x)(3y-1)$
$(3a+2b^2)(5a^2-3b)$
$(2x+1)(3x-5x^3)$
$(y-1)(3x+5y)$
$(7x-2)(7y+4)$

Factorising Quadratics

$(x+2)(x+6)$
$(x-2)(x-12)$
$(x-2)(x-2)$
$(x-3)(x+5)$
$(x-10)(x+2)$
$(x-5)(x+3)$
$(x-5)(x-2)$
$(x-5)(x-3)$
$(x+5)(x+12)$
$(x-3)(x+18)$
$(x-12)(x-13)$
$(3x+5)(x+1)$
$(4x-5)(x-1)$
$(5x+3)(x-2)$
$(3x-2)(x+1)$
$(2x+3)(2x+3)$
$(5x-3)(x+1)$
$(2x-1)(3x-4)$

Difference of Two Squares

$(x-3)(x+3)$
$(2x-6)(2x+6)$
$(3x^2y-z^4)(3x^2y+z^4)$
$(4x^3-1)(4x^3+1)$
$(11-10x^5)(11+10x^5)$

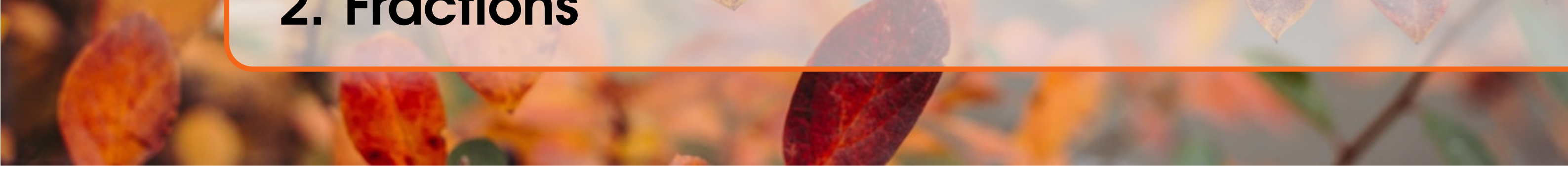
*Sum and Difference of Two Cubes

$(x-2)(x^2+2x+4)$
$(2x+6)(4x^2-12x+36)$
$(3x^2y-z^3)(9x^4y^2+3x^2yz^3+z^6)$
$(4x^2+1)(16x^4-4x^2+1)$
$(5-7x)(25+35x+49x^2)$

*Binomial Expansions

$x^7+7x^6y+21x^5y^2+35x^4y^3+35x^3y^4+21x^2y^5+7xy^6+y^7$
$256x^4+256x^3+96x^2+16x+1$
$27x^3+54x^2+36x+8$
$1024x^5-2560x^4+2560x^3-1280x^2+320x-32$
$16x^4-160x^3+600x^2-1000x+625$

126
$81,648$
$-20,412$
-30
512



2. Fractions

2.1 Some Terms and Prerequesites

When dealing with fractions we must first remember the mathematical terms for the "top" and "bottom" of the fraction.

Definition 2.1 For any fraction $\frac{p}{q}$, p is the **numerator** and q is the **denominator**.

For any fractions, we have the following rules.

Rule 2.2

1. Multiplying or dividing the numerator and denominator of a fraction by the same term does not change the value of a fraction, only how it looks. For example,

$$\frac{3}{4}=\frac{9}{12} \quad \text{(multiply by 3)}, \qquad \frac{15x}{10}=\frac{3x}{2} \quad \text{(divide by 5)}, \qquad \frac{5x}{2}=\frac{5x^2}{2x} \quad \text{(multiply by } x\text{)}.$$

2. If either the numerator or denominator of a fraction has more than one term, multiplying/dividing the numerator and denominator involves multiplying/dividing all terms. For example,

$$\frac{3+2x}{4}=\frac{9+6x}{12} \quad \text{(multiply by 3)}, \qquad \frac{15x}{10+5x}=\frac{3x}{2+x} \quad \text{(divide by 5)},$$
$$\frac{5x-3}{2+3x^2}=\frac{5x^2-3x}{2x+3x^3} \quad \text{(multiply by } x\text{)}, \qquad \frac{4x+2x^2}{5x^2-3x}=\frac{4+2x}{5x-3} \quad \text{(divide by } x\text{)}.$$

3. Fractions can be added/subtracted if they have the same denominator, in which case we add/subtract the numerators and the denominator remains unchanged. For example,

$$\frac{3}{5}+\frac{1}{5}=\frac{4}{5}, \qquad \frac{3+x}{4}+\frac{2+3x}{4}=\frac{5+4x}{4}, \qquad \frac{3}{4+x}-\frac{2}{4+x}=\frac{1}{4+x}.$$

4. A non-fraction can be turned into a fraction by giving it a denominator of 1. For example,

$$2=\frac{2}{1}, \qquad 5x=\frac{5x}{1}, \qquad 3+x=\frac{3+x}{1}.$$

2.2 Adding/Subtracting Fractions With Constant Denominator

We always add or subtract fractions in the following way.

1. Find a common denominator, i.e. a number that is a multiple of both denominators. Ideally this is the lowest common denominator/lowest common multiple.
2. For each fraction multiply both the numerator and denominator by the same number so that both fractions have denominator equal to the common denominator
3. Add or subtract them by adding or subtracting the numerators.

For example,

$$\begin{aligned}\frac{1}{6}+\frac{3}{4} &= \frac{2}{12}+\frac{9}{12}\\ &= \frac{11}{12},\\ \frac{3}{4}-\frac{1}{2} &= \frac{3}{4}-\frac{2}{4}\\ &= \frac{1}{4}.\end{aligned}$$

When adding or subtracting fractions with algebraic numerators we do the same thing.

Example 2.3 Write the following expression as a single fraction.

$$\frac{5+x}{3}+\frac{3+2x}{4}.$$

We first find the common denominator of 12. We then multiply the first fraction top and bottom by 4, and the second by 3. We then add the fractions, combining like terms.

$$\begin{aligned}\frac{5+x}{3}+\frac{3+2x}{4} &= \frac{20+4x}{12}+\frac{9+6x}{12}\\ &= \frac{20+4x+9+6x}{12}\\ &= \frac{29+10x}{12}.\end{aligned}$$

Example 2.4 Write the following expression as a single fraction.

$$\frac{3x+1}{6}-\frac{2x+5}{9}$$

$$\begin{aligned}\frac{3x+1}{6}-\frac{2x+5}{9}&=\frac{9x+3}{18}-\frac{4x+10}{18}\\&=\frac{9x+3-(4x+10)}{18}\\&=\frac{9x+3-4x-10}{18}\\&=\frac{5x-7}{18}.\end{aligned}$$

Notice that the minus sign affects both terms in the numerator of the second fraction.

If you find a common denominator that is not the lowest common denominator then you will find that you can cancel a factor from all terms in the numerator and denominator of the final fraction.

Example 2.5 — Example 2.4 Alternate Method. Write the following expression as a single fraction.

$$\frac{3x+1}{6}-\frac{2x+5}{9}$$

This time let's just multiply 6 and 9 to get a common denominator of 54. This means we end up multiplying each fraction top and bottom by the denominator of the other.

$$\begin{aligned}\frac{3x+1}{6}-\frac{2x+5}{9}&=\frac{27x+9}{54}-\frac{12x+30}{54}\\&=\frac{15x-21}{54}\\&=\frac{5x-7}{18}\end{aligned}$$

where we get the last equality by dividing each term in the fraction by 3.

Note 2.6 Instead of finding a lowest common denominator between two fractions, a common denominator can always be found by multiplying the two denominators, and two fractions can be given this common denominator by multiplying each fraction top and bottom by the denominator of the other. If fractions are added by finding a common denominator that is not the lowest common denominator, a common factor can be cancelled from all terms once the fractions are added/subtracted.

Question 2.7 Write the following expressions as a single fraction.

$$\frac{5+2x}{9}+\frac{3+x}{4}$$

$$\frac{x-3}{4}-\frac{2x+6}{5}$$

$$\frac{x+5}{2}-\frac{2x-3}{4}$$

$$\frac{5x-3}{5}+\frac{4-x}{3}$$

This logic can be applied to more than two fractions.

Example 2.8 Write the following expression as a single fraction.

$$\frac{2x+1}{5}-\frac{x-3}{2}-\frac{3x+4}{4}$$

The common denominator of 2,5 and 4 is 20.

$$\begin{aligned}\frac{2x+1}{5}-\frac{x-3}{2}-\frac{3x+4}{4} &= \frac{8x+4}{20}-\frac{10x-30}{20}-\frac{15x+20}{20}\\ &= \frac{8x+4-(10x-30)-(15x+20)}{20}\\ &= \frac{8x+4-10x+30-15x-20}{20}\\ &= \frac{-17x+14}{20}.\end{aligned}$$

Question 2.9 Write the following expressions as a single fraction.

$$\frac{2x-3}{4}+\frac{x+3}{6}+\frac{3x-5}{3}$$

$$\frac{5-x}{6}+\frac{3x-4}{8}-\frac{x-2}{4}$$

$$\frac{3x+1}{5}-\frac{x+3}{10}+\frac{5x+3}{2}$$

Finally, we can add non-fractions to fractions by artificially turning them into fractions with denominator 1.

Example 2.10 Write the following expression as a single fraction.

$$\frac{2x+1}{5}+4$$

$$\begin{aligned}\frac{2x+1}{5}+4 &= \frac{2x+1}{5}+\frac{4}{1}\\ &= \frac{2x+1}{5}+\frac{20}{5}\\ &= \frac{2x+21}{5}.\end{aligned}$$

Question 2.11 Write the following expressions as a single fraction.

$$\frac{3x+4}{2}+1$$

$$\frac{5x-2}{4}-3$$

$$\frac{2x-3}{5}+2x$$

$$\frac{5-x}{6}-3x$$

2.3 Adding/Subtracting Fractions With Algebraic Denominator

What if, instead, the algebraic term is in the denominator? For example, consider the expression

$$\frac{5}{3+x}+\frac{2}{4+3x}.$$

The idea of a lowest common denominator between $3+x$ and $4+3x$ may be daunting, but think back to Note 2.6. We can always gives two fractions a common denominator by multiplying one fraction by the denominator of the other.

Example 2.12 Write the following expression as a single fraction.

$$\frac{5}{3+x}+\frac{2}{4+3x}.$$

$$\begin{aligned}\frac{5}{3+x}+\frac{2}{4+3x} &= \frac{5(4+3x)}{(3+x)(4+3x)}+\frac{2(3+x)}{(4+3x)(3+x)}\\ &= \frac{20+15x}{(3+x)(4+3x)}+\frac{6+2x}{(3+x)(4+3x)}\\ &= \frac{26+17x}{(3+x)(4+3x)}.\end{aligned}$$

There is no real reason not to multiply out the brackets in the denominator, except that as we saw in Chapter 1 it's easier to expand brackets than to factorise. Therefore if this simplifying of fractions is part of a larger problem it may be better to leave the denominator factored unless it becomes clearly advantageous to expand the brackets.

Question 2.13 Write the following expressions as a single fraction.

$$\frac{9}{5+2x}+\frac{4}{3+x}$$

$$\frac{4}{x-3}-\frac{5}{2x+6}$$

$$\frac{2}{x+5}-\frac{4}{2x-3}$$

$$\frac{5}{5x-3}+\frac{3}{4-x}$$

Now that we have seen how to add or subtract two fractions we can tackle three fractions.

Example 2.14 Write the following expression as a single fraction.

$$\frac{3}{x+1}+\frac{4}{x+2}+\frac{5}{x+3}.$$

We will write these fractions so that they all have the denominator

$$(x+1)(x+2)(x+3),$$

the product of the three denominators. We will do this by multiplying each fraction by whatever factors its denominator is missing.

$$\begin{aligned}
&\frac{3}{x+1}+\frac{4}{x+2}+\frac{5}{x+3}\\
=&\frac{3(x+2)(x+3)}{(x+1)(x+2)(x+3)}+\frac{4(x+1)(x+3)}{(x+2)(x+1)(x+3)}+\frac{5(x+1)(x+2)}{(x+3)(x+1)(x+2)}\\
=&\frac{3x^2+15x+18}{(x+1)(x+2)(x+3)}+\frac{4x^2+16x+12}{(x+2)(x+1)(x+3)}+\frac{5x^2+15x+10}{(x+3)(x+1)(x+2)}\\
=&\frac{12x^2+46x+40}{(x+1)(x+2)(x+3)}.
\end{aligned}$$

Question 2.15 Write the following expressions as a single fraction.

$$\frac{5}{2x+1}+\frac{3}{x-4}+\frac{1}{x+3}$$

$$\frac{3}{2x-1}-\frac{5}{3x+1}-\frac{1}{x-2}$$

$$\frac{2}{x+4}+\frac{3}{2x-3}-\frac{3}{x-7}$$

In contrast to Example 2.14, consider the following example.

Example 2.16 Write the following expression as a single fraction.

$$\frac{3}{x-1}+\frac{4}{(x-1)(x+2)}-\frac{7}{x+2}.$$

It would be unwise to multiply each fraction top and bottom by the denominators of the other two when we can instead write each fraction with the denominator $(x-1)(x+2)$.

$$\begin{aligned}
&\frac{3}{x-1}+\frac{4}{(x-1)(x+2)}-\frac{7}{x+2}\\
=&\frac{3(x+2)}{(x-1)(x+2)}+\frac{4}{(x-1)(x+2)}-\frac{7(x-1)}{(x+2)(x-1)}\\
=&\frac{3x+6}{(x-1)(x+2)}+\frac{4}{(x-1)(x+2)}-\frac{7x-7}{(x+2)(x-1)}\\
=&\frac{-4x+17}{(x-1)(x+2)}.
\end{aligned}$$

Sometimes we may need to factorise before we realise this is a viable strategy.

Example 2.17 Write the following expression as a single fraction.

$$\frac{3}{x+3}+\frac{2}{x+4}+\frac{1}{x^2+7x+12}.$$

We first factorise the quadratic in the denominator of the final fraction, then combine them as in Example 2.16.

$$\begin{aligned}
\frac{3}{x+3}+\frac{2}{x+4}+\frac{1}{x^2+7x+12}&=\frac{3}{x+3}+\frac{2}{x+4}+\frac{1}{(x+3)(x+4)}\\
&=\frac{3(x+4)}{(x+3)(x+4)}+\frac{2(x+3)}{(x+4)(x+3)}+\frac{1}{(x+3)(x+4)}\\
&=\frac{3x+12}{(x+3)(x+4)}+\frac{2x+6}{(x+4)(x+3)}+\frac{1}{(x+3)(x+4)}\\
&=\frac{5x+19}{(x+3)(x+4)}.
\end{aligned}$$

Question 2.18 Write the following expressions as a single fraction.

$$\frac{5}{x+4}+\frac{2}{x+3}+\frac{1}{(x+4)(x+3)}$$

$$\frac{1}{(x+3)(x-2)}-\frac{2}{x+3}+\frac{3}{x-2}$$

$$\frac{1}{x-2}+\frac{5}{x^2-4}+\frac{2}{x+2}$$

In a similar way to Section 2.2 we can add fractions to non-fractions by giving the non-fraction a denominator of 1.

Example 2.19 Write the following expression as a single fraction.

$$\frac{3}{2+5x}+3$$

$$\begin{aligned}\frac{3}{2+5x}+3 &= \frac{3}{2+5x}+\frac{3}{1}\\ &= \frac{3}{2+5x}+\frac{3(2+5x)}{2+5x}\\ &= \frac{3}{2+5x}+\frac{6+15x}{2+5x}\\ &= \frac{9+15x}{2+5x}.\end{aligned}$$

Question 2.20 Write the following expressions as a single fraction.

$$\frac{9}{5+2x}+1$$

$$\frac{4}{x-3}-2x$$

$$\frac{2}{x+5}-3$$

$$\frac{5}{5x-3}+4x$$

There are two cases of linear algebraic denominators (denominators in the form $ax+b$) we haven't yet considered. The first is when one (or both) of the denominators is a multiple of x. When one of the denominators is a multiple of x this is treated the same as our previous work.

Example 2.21 Write the following expression as a single fraction.

$$\frac{3}{5+2x}+\frac{3}{4x}$$

$$\begin{aligned}\frac{3}{5+2x}+\frac{3}{4x} &= \frac{3(4x)}{(5+2x)(4x)}+\frac{3(5+2x)}{4x(5+2x)}\\ &= \frac{12x}{(5+2x)(4x)}+\frac{15+6x}{(5+2x)(4x)}\\ &= \frac{15+18x}{(5+2x)(4x)}.\end{aligned}$$

However if both denominators are multiples of x we can simply concentrate on the numbers as if they were arithmetic fractions.

Example 2.22 Write the following expression as a single fraction.

$$\frac{3}{4x} - \frac{1}{2x}$$

$$\begin{aligned}\frac{3}{4x} - \frac{1}{2x} &= \frac{3}{4x} - \frac{2}{4x}\\ &= \frac{1}{4x}.\end{aligned}$$

Question 2.23 Write the following expressions as a single fraction.

$$\frac{3}{4x+1} + \frac{5}{6x}$$

$$\frac{7}{2x} - \frac{3}{x}$$

$$\frac{3}{5x} - \frac{3}{4x}$$

$$\frac{3}{5x-7} - \frac{3}{4x}$$

The final case we haven't considered is the case where the denominator of one fraction is the negative of the denominator of the other.

Example 2.24 Write the following expression as a single fraction.

$$\frac{3}{5-2x} + \frac{7}{2x-5}.$$

In this case we can multiply one of the fractions top and bottom by -1 to make the denominators equal:

$$\begin{aligned}\frac{3}{5-2x} + \frac{7}{2x-5} &= \frac{3}{5-2x} + \frac{-7}{5-2x}\\ &= \frac{-4}{5-2x}.\end{aligned}$$

Example 2.25 Write the following expression as a single fraction.

$$\frac{2}{3x-1} - \frac{5}{1-3x}.$$

$$\frac{2}{3x-1}-\frac{5}{1-3x}=\frac{2}{3x-1}-\frac{-5}{3x-1}$$
$$=\frac{7}{3x-1}.$$

Question 2.26 Write the following expressions as a single fraction.

$$\frac{5}{x-3}+\frac{4}{3-x}$$
$$\frac{8}{2x-5}-\frac{5}{5-2x}$$
$$\frac{3}{2-x}+\frac{7}{x-2}$$
$$\frac{9}{6-4x}-\frac{5}{4x-6}$$

Finally, we can apply a similar logic when one denominator is constant and one is algebraic.

Example 2.27 Write the following expressions as a single fraction.

$$\frac{2}{x+2}+\frac{3}{4}$$

We can multiply $x+2$ and 4 to get a common denominator of $4(2+x)$.

$$\frac{2}{x+2}+\frac{x}{4}=\frac{2(4)}{4(x+2)}+\frac{3(x+2)}{4(x+2)}$$
$$=\frac{8}{4(x+2)}+\frac{3x+6}{4(x+2)}$$
$$=\frac{3x+14}{4(x+2)}.$$

Question 2.28 Write the following expressions as a single fraction.

$$\frac{3}{x-2}+\frac{3}{4}$$
$$\frac{5}{x+1}-\frac{4}{5}$$
$$\frac{3}{x}+\frac{5}{2}$$
$$\frac{2}{5x}+\frac{3}{10}$$

2.4 Simplifying Fractions, and Polynomial Long Division

Given a single fraction, it's possible that we can factorise and cancel some factors, overall simplifying the fraction.

Example 2.29 Factorise the numerator and denominator to simplify the following fraction.

$$\frac{x^2-1}{x^2+6x+5}$$

First we factorise the numerator and denominator, and then cancel the $(x+1)$ factor from both.

$$\frac{x^2-1}{x^2+6x+5} = \frac{(x-1)\cancel{(x+1)}}{(x+5)\cancel{(x+1)}}$$

$$= \frac{x-1}{x+5}.$$

Sometimes this results in the expression no longer being a fraction.

Example 2.30 Factorise the numerator and denominator to simplify the following fraction.

$$\frac{x^2+5x}{x+5}$$

$$\frac{x^2+5x}{x+5} = \frac{x\cancel{(x+5)}}{\cancel{x+5}}$$

$$= x.$$

Question 2.31 Possibly with the help of factoring, cancel common factors to simplify the following fractions.

$$\frac{3x+18}{6x+12}$$

$$\frac{x^2+11x+10}{x^2+7x+6}$$

$$\frac{x^3-5x^2}{2x^7-9x^4}$$

$$\frac{x^2+11x+24}{x+8}$$

Given a more complicated numerator we may not be able to factor it by previous methods. In this case we will use **polynomial long division**.

Example 2.32 Use polynomial long division to simplify the following fraction.

$$\frac{6x^3+9x^2+x+14}{x+2}.$$

We set up our long division as follows.

$$\begin{array}{r|l} x+2 & \overline{6x^3+9x^2+x+14} \end{array}$$

We then divide the largest power term on the outside by the largest power term on the inside, and place it above.

$$\begin{array}{r|l} & 6x^2 \\ x+2 & \overline{6x^3+9x^2+x+14} \end{array}$$

We then multiply the entire expression on the outside by the term we just found, and place the terms under the terms in the inside of the same power.

$$\begin{array}{r|l} & 6x^2 \\ x+2 & \overline{6x^3+9x^2+x+14} \\ & 6x^3+12x^2 \end{array}$$

We then subtract the bottom line from the top line, like when doing simultaneous equations.

$$\begin{array}{rrrrrrrr} & & 6x^2 & & & & & \\ \cline{2-8} x+2 & | & 6x^3 & + & 9x^2 & + & x & + 14 \\ & (-) & 6x^3 & + & 12x^2 & & & \\ \cline{3-5} & & & & -3x^2 & & & \end{array}$$

We then bring down the remaining terms.

$$\begin{array}{rrrrrrrr} & & 6x^2 & & & & & \\ \cline{2-8} x+2 & | & 6x^3 & + & 9x^2 & + & x & + 14 \\ & (-) & 6x^3 & + & 12x^2 & & & \\ \cline{3-5} & & & & -3x^2 & + & x & + 14 \end{array}$$

We then repeat. This process can be seen as doing the operations $+-\times\div$ in reverse order. We **divided** the $6x^2$ by x. We **multiplied** the $x+2$ by $6x^2$. We **subtracted** the bottom line from the top, and we **added** the other terms to the last line. Repeating this process,

$$\begin{array}{rrrrrrrr} & & 6x^2 & - & 3x & & & \\ \cline{2-8} x+2 & | & 6x^3 & + & 9x^2 & + & x & + 14 \\ & (-) & 6x^3 & + & 12x^2 & & & \\ \cline{3-5} & & & & -3x^2 & + & x & + 14 \\ & & & (-) & -3x^2 & - & 6x & \\ \cline{5-7} & & & & & & 7x & + 14 \end{array}$$

$$
\begin{array}{rrrrrrrrrrr}
 & & & & 6x^2 & - & 3x & + & 7 & & \\
\cline{4-11}
x & + & 2 & | & 6x^3 & + & 9x^2 & + & x & + & 14 \\
 & & & (-) & 6x^3 & + & 12x^2 & & & & \\
\cline{5-7}
 & & & & & & -3x^2 & + & x & + & 14 \\
 & & & & & (-) & -3x^2 & - & 6x & & \\
\cline{7-9}
 & & & & & & & & 7x & + & 14 \\
 & & & & & & & (-) & 7x & + & 14 \\
\cline{9-11}
 & & & & & & & & & & 0
\end{array}
$$

Therefore

$$\frac{6x^3+9x^2+x+14}{x+2} = 6x^2-3x+7.$$

Note 2.33 The 0 at the end of Example 2.32 is the remainder. In the vast majority of questions related to the Leaving Cert syllabus the remainder will be 0, including those in Question 2.35. Moreover, when we apply polynomial long division to other problems, such as solving cubic equations in Section 3.4 later in the year, we will have reasons to expect a remainder of 0. Therefore if you don't get a remainder of 0 you know you made a mistake. We will briefly discuss polynomial long division with non-zero remainder in Section 2.5.

Remember that when the coefficient of the leading term of the divisor is not 1 that it needs to be included in the division.

Example 2.34 Use polynomial long division to simplify the following fraction.

$$\frac{2x^3-15x^2+4x+6}{2x+1}$$

$$
\begin{array}{rrrrrrrrrrr}
 & & & & x^2 & - & 8x & + & 6 & & \\
\cline{4-11}
2x & + & 1 & | & 2x^3 & - & 15x^2 & + & 4x & + & 6 \\
 & & & (-) & 2x^3 & + & x^2 & & & & \\
\cline{5-7}
 & & & & & & -16x^2 & + & 4x & + & 6 \\
 & & & & & (-) & -16x^2 & - & 8x & & \\
\cline{7-9}
 & & & & & & & & 12x & + & 6 \\
 & & & & & & & (-) & 12x & + & 6 \\
\cline{9-11}
 & & & & & & & & & & 0
\end{array}
$$

Therefore

$$\frac{2x^3-15x^2+4x+6}{2x+1} = x^2-8x+6.$$

Question 2.35 Use polynomial long division to simplify the following fractions.

$$\frac{8x^3+19x^2-7x+24}{x+3}$$

$$\frac{3x^3-17x^2+19x-36}{x-4}$$

$$\frac{6x^3+5x^2+9x+4}{2x+1}$$

We can also use polynomial long division to divide by quadratics.

Example 2.36 Use polynomial long division to simplify the following fraction.

$$\frac{x^3+6x^2+11x+6}{x^2+5x+6},$$

$$x^2 + 5x + 6 \;\big|\; \overline{x^3 + 6x^2 + 11x + 6}$$

Our steps are similar.

$$\begin{array}{rrrrrrrrrr}
 & & & & & x & & & & \\ \cline{5-10}
x^2+5x+6 & & & & | & x^3 & + & 6x^2 & + & 11x \;+\; 6 \\
 & & & & (-) & x^3 & + & 5x^2 & + & 6x \\ \cline{6-10}
 & & & & & & & x^2 & + & 5x \;+\; 6
\end{array}$$

$$\begin{array}{rrrrrrrrrr}
 & & & & & x & + & 1 & & \\ \cline{5-10}
x^2+5x+6 & & & & | & x^3 & + & 6x^2 & + & 11x \;+\; 6 \\
 & & & & (-) & x^3 & + & 5x^2 & + & 6x \\ \cline{6-10}
 & & & & & & & x^2 & + & 5x \;+\; 6 \\
 & & & & & & (-) & x^2 & + & 5x \;+\; 6 \\ \cline{8-10}
 & & & & & & & & & 0
\end{array}$$

Therefore

$$\frac{x^3+6x^2+11x+6}{x^2+5x+6} = x+1.$$

Question 2.37 Use polynomial long division to simplify the following fraction.

$$\frac{x^3+6x^2+11x+6}{x^2+3x+2}$$

$$\frac{2x^3-x^2-9x-4}{x^2-x-4}$$

If we have a missing term, we should place it in the setup anyway so that terms are not out of place.

Example 2.38 Use polynomial long division to simplify the following fraction.

$$\frac{2x^3 - 15x^2 + 4}{2x + 1}$$

We set up our long division as follows, adding a $0x$ so that each column contains the same powers of x.

$$
\begin{array}{rrrrrrrrrrrr}
 & & & & x^2 & - & 8x & + & 4 & & \\ \cline{4-11}
2x & + & 1 & \Big| & 2x^3 & - & 15x^2 & + & 0x & + & 4 \\
 & & & (-) & 2x^3 & + & x^2 & & & & \\ \cline{5-7}
 & & & & & & -16x^2 & + & 0x & + & 4 \\
 & & & & & (-) & -16x^2 & - & 8x & & \\ \cline{7-9}
 & & & & & & & & 8x & + & 4 \\
 & & & & & & & (-) & 8x & + & 4 \\ \cline{9-11}
 & & & & & & & & & & 0
\end{array}
$$

Therefore

$$\frac{2x^3 - 15x^2 + 4}{2x + 1} = x^2 - 8x + 54.$$

Question 2.39 Use polynomial long division to simplify

$$\frac{x^3 + 4x - 5}{x - 1}$$

$$\frac{x^3 - 3x^2 + 20}{x + 2}$$

$$\frac{x^3 + 7x^2 + 12x}{x + 3}$$

2.5 *Polynomial Long Division with Non-Zero Remainders

What if there is a non-zero remainder? In this case the fraction can be written as a non-fraction and a fraction. Alternatively, the numerator of the fraction can be written as a multiple of the denominator and a leftover term. For example, we know that if we divide 45 by 7, we get 6 with a remainder of 3. This means two things.

$$\frac{45}{7} = 6 + \frac{3}{7} \qquad \text{and} \qquad 45 = 6(7) + 3.$$

Note that we can get from equation to the other by multiplying/dividing by 7.

The same is true of polynomial long division resulting in a remainder.

Theorem 2.40 — Polynomial Long Division with Remainder. If dividing $p(x)$ by $q(x)$ gives

$f(x)$ with a remainder of r, then

$$\frac{p(x)}{q(x)} = f(x) + \frac{r}{q(x)} \qquad \text{and} \qquad p(x) = f(x)q(x) + r.$$

Example 2.41 Use polynomial long division to find the remainder when $6x^3+9x^2+x+20$ is divided by $x+2$, and use it to write

$$\frac{6x^3+9x^2+x+20}{x+2}$$

in the form

$$\text{Quadratic} + \frac{\text{Constant}}{x+2}.$$

The polynomial long division is shown below.

$$\begin{array}{cccccccccc}
 & & & & 6x^2 & - & 3x & + & 7 & & \\
\cline{4-11}
x & + & 2 & | & 6x^3 & + & 9x^2 & + & x & + & 20 \\
 & & & (-) & 6x^3 & + & 12x^2 & & & & \\
\cline{5-7}
 & & & & & & -3x^2 & + & x & + & 20 \\
 & & & & & (-) & -3x^2 & - & 6x & & \\
\cline{7-9}
 & & & & & & & & 7x & + & 20 \\
 & & & & & & & (-) & 7x & + & 14 \\
\cline{9-11}
 & & & & & & & & & & 6
\end{array}$$

Therefore we have a remainder of 6. This means that

$$\frac{6x^3+9x^2+x+20}{x+2} = 6x^2-3x+7+\frac{6}{x+2}.$$

Question 2.42 If we know that

$$\begin{array}{ccccccccccc}
 & & & & 24x^2 & + & 55x & + & 25 & & \\
\cline{4-11}
x & - & 2 & | & 24x^3 & + & 7x^2 & - & 85x & - & 47 \\
 & & & (-) & 24x^3 & - & 48x^2 & & & & \\
\cline{5-7}
 & & & & & & 55x^2 & - & 85x & - & 47 \\
 & & & & & (-) & 55x^2 & - & 110x & & \\
\cline{7-9}
 & & & & & & & & 25x & - & 47 \\
 & & & & & & & (-) & 25x & - & 50 \\
\cline{9-11}
 & & & & & & & & & & 3
\end{array}$$

write out

$$\frac{24x^3+7x^2-85x-47}{x-2}$$

in the form

$$\text{Quadratic} + \frac{\text{Constant}}{x-2}.$$

Question 2.43 If we know that

$$
\begin{array}{rrrrrrrrrr}
 & & & 2x^2 & + & 11x & + & 14 & & \\
\cline{3-10}
x & - & 2 \;|\; & 2x^3 & + & 7x^2 & - & 8x & - & 4 \\
 & & (-) & 2x^3 & - & 4x^2 & & & & \\
\cline{4-6}
 & & & & & 11x^2 & - & 8x & - & 4 \\
 & & & & (-) & 11x^2 & - & 22x & & \\
\cline{6-8}
 & & & & & & & 14x & - & 4 \\
 & & & & & & (-) & 14x & - & 28 \\
\cline{8-10}
 & & & & & & & & & 24
\end{array}
$$

write out

$$\frac{2x^3+7x^2-8x-4}{x-2}$$

in the form

$$\text{Quadratic} + \frac{\text{Constant}}{x-2}.$$

2.6 *Complex Fractions

How do we get rid of fractions within fractions (known as complex fractions)? For example, we should be able to write the fraction

$$\frac{\frac{3}{x}+4}{7x-5}$$

in a simpler way. We will do this by multiplying all terms by the denominator of the fraction within the fraction.

Example 2.44 Write the fraction

$$\frac{\frac{3}{x}+4}{7x-5}$$

as a fraction where there are no fractions in the numerator or denominator.

Multiplying all terms in the fraction top and bottom by x,

$$\frac{\frac{3}{x}+4}{7x-5} = \frac{3+4x}{7x^2-5x}.$$

Remember that if we multiply a fraction by its denominator the denominator disappears and we're left with just the numerator.

If there is more than one fraction inside the main fraction, we multiply by the common denominator of the fractions.

Example 2.45 Write the fraction

$$\frac{5x+\frac{1}{3}}{\frac{x}{4}+3}$$

as a fraction where there are no fractions in the numerator or denominator.

The common denominator of 3 and 4 is 12, and so multiplying the fraction top and bottom by 12,

$$\begin{aligned}\frac{5x+\frac{1}{3}}{\frac{x}{4}+3} &= \frac{60x+\frac{12}{3}}{\frac{12x}{4}+36} \\ &= \frac{60x+4}{3x+12}.\end{aligned}$$

Example 2.46 Write the fraction

$$\frac{\frac{4}{x}-3}{\frac{x}{2}+5}.$$

as a fraction where there are no fractions in the numerator or denominator.

The common denominator of the fractions is $2x$, so multiplying each term by $2x$ we have

$$\begin{aligned}\frac{\frac{4}{x}-3}{\frac{x}{2}+5} &= \frac{\frac{8x}{x}-6x}{\frac{2x^2}{2}+10x} \\ &= \frac{8-6x}{x^2+10x}.\end{aligned}$$

Question 2.47 Write each of the following complex fractions as fractions where there are no fractions in the numerator or denominator.

$$\frac{\frac{x}{5}-3}{\frac{3}{x+1}+3}$$

$$\frac{1}{\frac{1}{p}+\frac{1}{q}}$$

$$\frac{3-\frac{x}{x+1}}{\frac{5}{x+1}+2}$$

2.7 Summary

Similar to Chapter 1, simplifying and combining fractions are not usually standalone questions on Leaving Cert papers in Higher or Ordinary Level, with the except of polynomial long division which does occasionally appear by itself. Usually however, it too is part of a larger problem related to the study of cubics which we will study in Chapter 3.

2.8 Homework

Adding/Subtracting Fractions With Constant Denominator

1. Write the following expressions as a single fraction.

   (a) $\frac{3+x}{4}+\frac{5+2x}{9}$

   (b) $\frac{5-2x}{3}-\frac{3+4x}{6}$

   (c) $\frac{5-x}{9}-\frac{3-2x}{6}$

   (d) $\frac{3x+7}{5}-\frac{2x-1}{10}$

Adding/Subtracting Fractions With Algebraic Denominator

2. Write the following expressions as a single fraction.

   (a) $\frac{4}{3+x}+\frac{9}{5+2x}$

   (b) $\frac{3}{5-2x}-\frac{6}{3+4x}$

   (c) $\frac{9}{5-x}-\frac{6}{3-2x}$

   (d) $\frac{5}{3x+7}-\frac{10}{2x-1}$

   (e) $\frac{5}{2x}+\frac{3}{4x}$

   (f) $\frac{3}{8x}-\frac{5}{2-x}$

   (g) $\frac{5}{5-2x}+\frac{3}{2x-5}$

   (h) $\frac{3}{x-7}-\frac{4}{7-x}$

Simplifying Fractions, and Polynomial Long Division

3. Possibly with the help of factoring, cancel common factors to simplify the following fractions.

   (a) $\frac{5x^2-3x^3}{5x+7x^2}$

   (b) $\frac{3x^2-6x}{x^2-5x+6}$

   (c) $\frac{x^2-9}{x^2-7x+12}$

   (d) $\frac{x+5}{x^2+7x+10}$

   (e) $\frac{x^3-8}{x-2}$

   (f) $\frac{3x+9}{6}$

4. Use polynomial long division to simplify the following fractions.

   (a) $\frac{x^3+5x^2+3x-6}{x+2}$

   (b) $\frac{2x^3-7x^2+4x-3}{x-3}$

   (c) $\frac{2x^3+13x^2+7x-12}{2x+3}$

5. Use polynomial long division to simplify the following fractions.

   (a) $\frac{x^3+5x^2+3x-6}{x^2+3x-3}$

   (b) $\frac{2x^3-7x^2+4x-3}{2x^2-x+1}$

   (c) $\frac{x^3+3x^2-14x+8}{x^2+5x-4}$

6. Use polynomial long division to simplify the following fractions.

   (a) $\frac{x^3+4x^2-24}{x-2}$

   (b) $\frac{2x^3+3x+63}{x+3}$

   (c) $\frac{2x^3-9x^2+25}{2x-5}$

*Polynomial Long Division with Non-Zero Remainders

7. Use polynomial long division to write each of the following fractions in the form Quadratic $+\frac{\text{Constant}}{\text{Linear}}$.

   (a) $\frac{x^3+7x^2+3x+4}{x+2}$

   (b) $\frac{2x^3-5x^2+7x-3}{x-3}$

   (c) $\frac{3x^3+9x^2-4x-1}{x-2}$

*Complex Fractions

8. Write each of the following complex fractions as one where there are no fractions in the numerator or denominator.

   (a) $\frac{a-\frac{1}{b}}{a+\frac{1}{b}}$

(b) $\frac{3-\frac{5}{x+1}}{\frac{5}{2}-3x}$

(c) $\frac{2+\frac{1}{(x-1)(x+1)}}{3+\frac{5}{x-1}}$

2.9 Homework Solutions

Adding/Subtracting Fractions With Constant Denominator

1. (a) $\frac{17x+47}{36}$
   (b) $\frac{7-8x}{6}$
   (c) $\frac{4x+1}{18}$
   (d) $\frac{4x+15}{10}$

Adding/Subtracting Fractions With Algebraic Denominator

2. (a) $\frac{17x+47}{(3+x)(5+2x)}$
   (b) $\frac{24x-21}{(5-2x)(3+4x)}$
   (c) $\frac{-12x-3}{(5-x)(3-2x)}$
   (d) $\frac{-20x-75}{(3x+7)(2x-1)}$
   (e) $\frac{13}{4x}$
   (f) $\frac{6-43x}{8x(x-2)}$
   (g) $\frac{2}{5-2x}$
   (h) $\frac{7}{x-7}$

Simplifying Fractions, and Polynomial Long Division

3. (a) $\frac{5x-3x^2}{5+7x}$
   (b) $\frac{3x}{x-3}$
   (c) $\frac{x+3}{x-4}$
   (d) $\frac{1}{x+2}$
   (e) x^2+2x+4
   (f) $\frac{x+3}{2}$
4. (a) x^2+3x-3
   (b) $2x^2-x+1$
   (c) x^2+5x-4
5. (a) $x+2$
   (b) $x-3$
   (c) $x-2$
6. (a) $x^2+6x+12$
   (b) $2x^2-6x+21$
   (c) x^2-2x-5

*Polynomial Long Division with Non-Zero Remainders

7. (a) $\frac{x^3+7x^2+3x+4}{x+2} = x^2+5x-7+\frac{18}{x+2}$
   (b) $\frac{2x^3-5x^2+7x-3}{x-3} = 2x^2+x+10+\frac{27}{x-3}$
   (c) $\frac{3x^3+9x^2-4x-1}{x-2} = 3x^2+15x+26+\frac{51}{x-2}$

*Complex Fractions

8. (a) $\frac{ab-1}{ab+1}$
   (b) $\frac{4-6x}{6x^2+x-5}$
   (c) $\frac{2x^2-1}{3x^2+5x+2}$

2.10 Revision

Adding/Subtracting Fractions

Write the following expressions as a single fraction.

$$\frac{5x+2}{3}+\frac{3x+1}{4}$$

$$\frac{3x-4}{7}-\frac{2x+1}{3}$$

$$\frac{4-3x}{6}-\frac{3-x}{4}$$

$$\frac{3}{x+2}+\frac{5}{2x+3}$$

$$\frac{5}{x-1}+\frac{3}{x-5}$$

$$\frac{7}{3-2x}-\frac{4}{x+2}$$

$$\frac{5}{2x+1}+\frac{3}{4}$$

$$\frac{3}{2x-1}-7$$

$$\frac{5}{3x}+x$$

$$\frac{4}{2x+1}+3+x$$

$$\frac{5}{x+1}+\frac{3}{x-1}+\frac{4}{x+3}$$

$$\frac{4}{x-2}-\frac{3}{x+5}+\frac{1}{x-3}$$

$$\frac{3}{x+1}+\frac{4}{x-1}+\frac{5}{(x+1)(x-1)}$$

$$\frac{5}{x+1}-\frac{3}{x+3}+\frac{2}{x+1}.$$

Simplifying Fractions, and Polynomial Long Division

Use factorising or direct cancellation to simplify the following fractions as much as possible.

$$\frac{4+6x}{8}$$

$$\frac{3x+5x^2}{2x}$$

$$\frac{x^3-8}{x-2}$$

$$\frac{x^2+3x+2}{x^2+8x+12}$$

$$\frac{x^2-9}{x^2-7x+12}$$

$$\frac{x+4}{x^3+64}$$

$$\frac{x^2+7x+10}{x+2}.$$

Use polynomial long division to simplify the following fractions. (Hint: You shouldn't get a remainder).

$$\frac{x^3+7x^2+14x+6}{x+3}$$

$$\frac{3x^3-11x^2+12x-4}{x-2}$$

$$\frac{4x^3+16x^2+7x-20}{2x+5}$$

$$\frac{6x^3-23x^2+23x-4}{3x-4}$$

$$\frac{x^3+3x^2-4}{x-1}$$

$$\frac{x^3-7x^2+12x}{x-3}$$

$$\frac{x^3+7x^2+14x+6}{x^2+4x+2}$$

$$\frac{3x^3-11x^2+12x-4}{3x^2-5x+2}$$

*Polynomial Long Division with Non-Zero Remainders

Use polynomial long division to write the following fractions in the form

$$\text{Quadratic} + \frac{\text{Constant}}{\text{Linear}}$$

(Hint: This time you should get a remainder).

$$\frac{3x^3+4x^2+5x+1}{x-3}$$

$$\frac{x^3-7x^2+9x-4}{x+1}$$

*Complex Fractions

Write each of the following complex fractions as fractions where there are no fractions in the numerator or denominator.

$$\frac{\frac{3}{x}+4}{4x+3}$$

$$\frac{\frac{5}{2}+x}{3-2x}$$

$$\frac{\frac{3}{x}+\frac{5}{2}}{4+x}$$

$$\frac{2+\frac{3}{x+2}}{\frac{x}{3}-4}$$

$$\frac{a-\frac{p}{q}}{b-\frac{1}{pq}}$$

$$\frac{\frac{3}{x-2}+4}{3-\frac{x}{x-2}}$$

$$\frac{\frac{5}{(x-1)(x-2)}+3}{\frac{4}{x-1}-\frac{3}{x-2}}$$

2.11 Revision Solutions

Adding/Subtracting Fractions

$$\frac{29x+11}{12}$$

$$\frac{-5x-19}{21}$$

$$\frac{-3x-1}{12}$$

$$\frac{11x+19}{(x+2)(2x+3)}$$

$$\frac{8x-28}{(x-5)(x-1)}$$

$$\frac{15x+2}{(3-2x)(x+2)}$$

$$\frac{23+6x}{4(2x+1)}$$

$$\frac{10-14x}{2x-1}$$

$$\frac{5+3x^2}{3x}$$

$$\frac{7+7x+2x^2}{2x+1}$$

$$\frac{12x^2+22x-10}{(x-1)(x+1)(x+3)}$$

$$\frac{2x^2+26x-88}{(x-3)(x-2)(x+5)}$$

$$\frac{7x+6}{(x-1)(x+1)}$$

$$\frac{4x+18}{(x+1)(x+3)}$$

Simplifying Fractions, and Polynomial Long Division

$$\frac{2+3x}{4}$$

$$\frac{3+5x}{2}$$

$$x^2+2x+4$$

$$\frac{x+1}{x+6}$$

$$\frac{x+3}{x-4}$$

$$\frac{1}{x^2-4x+16}$$

$$x+5$$

$$x^2+4x+2$$

$$3x^2-5x+2$$

$$2x^2+3x-4$$

$$2x^2-5x+1$$

$$x^2+4x+4$$

$$x^2-4x$$

$$x+3$$

$$x-2$$

*Polynomial Long Division with Non-Zero Remainders

$$3x^2+13x+44+\frac{133}{x-3}$$

$$x^2-8x+17+\frac{-21}{x+1}$$

*Complex Fractions

$$\frac{3+4x}{4x^2+3x}$$

$$\frac{5+2x}{6-4x}$$

$$\frac{6+5x}{8x+2x^2}$$

$$\frac{6x+21}{x^2-10x-24}$$

$$\frac{apq-p^2}{bpq-1}$$

$$\frac{4x-5}{2x-6}$$

$$\frac{3x^2-9x+11}{x-5}$$